Math 6172 Answers to Problems in Test III Spring 2003

1. (a) $\dfrac{50 \cdot 49 \cdot 48}{3 \cdot 2 \cdot 1} = 19600.$

   (b) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$

   (c) $\dfrac{\dfrac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} \cdot \dfrac{6 \cdot 5}{2. \cdot 1}}{\dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \dfrac{5}{21}.$

   (d)
$$P(A \cap B \cap C) = P(C|A \cap B) P(A \cap B) = P(C|A \cap B) P(B|A) P(A).$$

   (e)
$$0.9^4 + 4 \cdot 0.9^3 \cdot 0.1 = 0.9477.$$

   (f) Take $n + m - 1$ card. The number of male cards could be
$$\underbrace{0, 1, \cdots, x, \cdots, n-1,}_{\substack{\text{satisfying our} \\ \text{requirement}}} \quad \underbrace{n, \cdots, n+m-1.}_{\substack{\text{not satisfying} \\ \text{our requirement}}}$$
When the number of male cards equals x, the probability is
$$\binom{n+m-1}{x} 0.2^x 0.8^{n+m-1-x}.$$
Therefore the total probability is
$$\sum_{x=0}^{n-1} \binom{n+m-1}{x} 0.2^x \cdot 0.8^{n+m-1-x}.$$

2. (a)
$$\begin{aligned} E\left[x^2\right] &= n^2p^2 + np(1-p), \\ Var[x] &= E\left[x^2\right] - E^2[x] = np(1-p). \end{aligned}$$

   (b)
$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \mu \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \mu. \end{aligned}$$

3. (a)

$$\begin{aligned} f_1(x) &= \int_0^1 (x+y)\,dy = x + \frac{1}{2}, \\ f_2(y) &= \int_0^1 (x+y)\,dx = \frac{1}{2} + y, \\ g_1(x) &= \int_0^1 \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)dy = \left(x+\frac{1}{2}\right)\left(\frac{y^2}{2}+\frac{1}{2}g\right)\Big|_0^1 = x + \frac{1}{2}, \\ g_2(y) &= y + \frac{1}{2}. \end{aligned}$$

Therefore

$$f_1(x) = g_1(x), \quad f_2(y) = g_2(y).$$

(b)

$$E[X] = 15000, \quad \sigma = 2.$$

4.

$$\mu = \frac{\sum_{i=1}^{k} x_i}{k}.$$

5. (a)

$$CONF_{0.99}(49.96 \le \mu \le 50.64).$$

(b)

$$CONF_{0.95}\left(0.265 \le \sigma^2 \le 1.867\right).$$

6. (a)

$$P\left(\frac{\bar{x}-\mu}{S/\sqrt{n}} \le c\right) = P\left(\frac{\bar{x}-\mu}{S/\sqrt{n}} \ge -c\right) = 0.95 \quad \Rightarrow \quad c = -1.66, \quad (n = 99)$$

$$\frac{34-40}{\sqrt{\frac{900}{100}}} = \frac{-6}{3} = -2 < -1.66, \quad \text{Reject}$$

(b)

$$P\left(\frac{\bar{x}-\mu}{S/\sqrt{n}} \ge -c\right) = 0.99 \Rightarrow c = -2.36 \quad (n = 99)$$

$$\frac{34-40}{3} = -2 > -2.36, \quad \text{Do not reject.}$$

7. From the data given, we can have

x	b_j	e_j	$\frac{(b_j-e_j)^2}{e_j}$
1	10	10	0
2	13	10	$\frac{9}{10}$
3	9	10	$\frac{1}{10}$
4	11	10	$\frac{1}{10}$
5	9	10	$\frac{1}{10}$
6	8	10	$\frac{4}{10}$
			$x_0^2 = \frac{16}{10} = 1.6$

$$k = 6, \qquad r = 0. \qquad k - r - 1 = 5, \qquad c = 11.07 > 1.6.$$

Do not reject.

8. From the data given, we can have

x_i	y_i	x_j^2	$x_j y_j$
32	0.337	1024	10.784
50	0.345	2500	17.25
100	0.365	10000	36.50
150	0.380	22500	57.00
212	0.395	44944	83.74
544	1.822	80968	205.274

$$\begin{aligned} k_1 &= \frac{\sum x_j y_j - \frac{1}{n}\sum x_j \sum y_j}{\sum x_j^2 - \frac{1}{n}\left(\sum x_j\right)^2} = \frac{205.274 - \frac{1}{5}544 \cdot 1.822}{80968 - \frac{1}{5}(544)^2} = \frac{205.274 - 198.2336}{80968 - 59187.2} \\ &= \frac{7.0404}{21780.8} = 0.0003232. \end{aligned}$$

$$k_0 = \frac{\sum y_i - k_1 \sum x_j}{n} = \frac{1.822 - 0.0003232 \cdot 544}{5} = 0.3292358.$$

$$y(66) = 0.3292358 + 66 \times 0.0003232 = 0.350567.$$