”Self-Checking Test for Calculus and Linear Algebra”

Name :________________________

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Show the details of your work !!

1. (a) Let $\xi = \dfrac{S}{S+P_m}$, $\tau = T - t$ and $V(S,t) = (S+P_m)\overline{V}(\xi,\tau)$, where P_m is a positive constant. Show that in terms of $\xi, \overline{V}, \dfrac{\partial \overline{V}}{\partial \tau}, \dfrac{\partial \overline{V}}{\partial \xi}$, and $\dfrac{\partial^2 \overline{V}}{\partial \xi^2}$, for $\dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial S}, \dfrac{\partial^2 V}{\partial S^2}$, there are the following expressions:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= -\frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial \tau}, \\
\frac{\partial V}{\partial S} &= \overline{V}(\xi,\tau) + (1-\xi)\frac{\partial \overline{V}}{\partial \xi}, \\
\frac{\partial^2 V}{\partial S^2} &= \frac{(1-\xi)^3}{P_m}\frac{\partial^2 \overline{V}}{\partial \xi^2}.
\end{aligned}$$

   (b) Consider a function $V(Z_1, Z_2, Z_3)$. Let

$$\begin{cases}
\xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}} \equiv \xi_1(Z_1), \\[2ex]
\xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}} \equiv \xi_2(Z_1, Z_2), \\[2ex]
\xi_3 = \dfrac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}} \equiv \xi_3(Z_2, Z_3),
\end{cases}$$

where $Z_{1,l}$, $Z_{2,l}$, and $Z_{3,l}$ are constants. These relations also give the relations $Z_1 = Z_1(\xi_1)$, $Z_2 = Z_2(\xi_1,\xi_2)$, and $Z_3 = Z_3(\xi_1,\xi_2,\xi_3)$ implicitly. We define $\overline{V}(\xi_1,\xi_2,\xi_3) = V(Z_1(\xi_1), Z_2(\xi_1,\xi_2), Z_3(\xi_1,\xi_2,\xi_3))$ and clearly for $V(Z_1,Z_2,Z_3)$ we have the following expression:

$$V(Z_1,Z_2,Z_3) = \overline{V}(\xi_1(Z_1), \xi_2(Z_1,Z_2), \xi_3(Z_2,Z_3)).$$

Express

$$\frac{\partial V}{\partial Z_1}, \frac{\partial V}{\partial Z_2}, \frac{\partial V}{\partial Z_3}, \frac{\partial^2 V}{\partial Z_1^2}, \frac{\partial^2 V}{\partial Z_2^2}, \frac{\partial^2 V}{\partial Z_3^2}, \frac{\partial^2 V}{\partial Z_1 \partial Z_2}, \frac{\partial^2 V}{\partial Z_2 \partial Z_3}, \frac{\partial^2 V}{\partial Z_1 \partial Z_3}$$

as linear functions of

$$\frac{\partial \overline{V}}{\partial \xi_1}, \frac{\partial \overline{V}}{\partial \xi_2}, \frac{\partial \overline{V}}{\partial \xi_3}, \frac{\partial^2 \overline{V}}{\partial \xi_1^2}, \frac{\partial^2 \overline{V}}{\partial \xi_2^2}, \frac{\partial^2 \overline{V}}{\partial \xi_3^2}, \frac{\partial^2 \overline{V}}{\partial \xi_1 \partial \xi_2}, \frac{\partial^2 \overline{V}}{\partial \xi_2 \partial \xi_3}, \frac{\partial^2 \overline{V}}{\partial \xi_1 \partial \xi_3}.$$

2. $G(S)$ is defined by

$$G(S) = \frac{1}{\sqrt{2\pi}bS} e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2},$$

where a and b are positive numbers. Show that

(a) for any real number n

$$\int_0^c S^n G(S)dS = a^n e^{(n^2-n)b^2/2} N\left(\frac{\ln(c/a)+b^2/2}{b} - nb\right);$$

(b)

$$\int_c^\infty \ln S\, G\,(S)\, dS$$
$$= \frac{b}{\sqrt{2\pi}} e^{-\left[\ln(c/a)+b^2/2\right]^2/2b^2} + \left(\ln a - b^2/2\right) N\left(-\frac{\ln\,(c/a)+b^2/2}{b}\right),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi.$$

(Hint: Use the substitution $\eta(S) = \frac{\ln(S/a)+b^2/2}{b}$.)

3. (a) Show that

$$\phi(\mathbf{x}_0; \mathbf{x}, \tau) = \frac{1}{(4\pi\tau)^{n/2}} e^{-\sum\limits_{i=1}^{n} (x_i - x_{i0})^2/(4\tau)}$$

is a solution to

$$\frac{\partial \phi}{\partial \tau} = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}, \quad -\infty < \mathbf{x} < \infty, \quad 0 \le \tau,$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{pmatrix}$$

and $-\infty < \mathbf{x} < \infty$ means

$$-\infty < x_i < \infty, \quad i = 1, 2, \cdots, n.$$

(b) Show that the function $\phi(\mathbf{x}_0; \mathbf{x}, \tau)$ satisfies the conditions

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\mathbf{x}_0; \mathbf{x}, \tau) dx_{10} dx_{20} \cdots dx_{n0} = 1$$

and

$$\lim_{\tau \to 0} \phi(\mathbf{x}_0; \mathbf{x}, \tau) = \begin{cases} \infty, & \text{at } \mathbf{x} = \mathbf{x}_0, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$\lim_{\tau \to 0} \phi(\mathbf{x}_0; \mathbf{x}, \tau) = \delta(\mathbf{x} - \mathbf{x}_0).$$

4. Suppose that $\varphi_1(\eta)$ and $\varphi_2(\eta)$ are defined for $\eta \in [0,1]$ and $\eta \in [1,\infty)$, respectively, and $\varphi_1(1) = \varphi_2(1)$ holds. Assume that

$$\frac{d\varphi_1(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \frac{d\varphi_2(1/\eta)}{d\eta}$$

and

$$\varphi_2(\eta) = \max(\eta - \beta, 0), \quad 1 \le \eta \quad \text{with} \quad \beta > 1.$$

Find the function $\varphi_1(\eta)$ for $\eta \in [0,1]$ if $r \neq D_0$.

5. The function $G(S', T; S, t)$ is defined by

$$\begin{aligned} & G(S', T; S, t) \\ = & \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} e^{-\left[\ln(S'/S) - \left(r - D_0 - \sigma^2/2\right)(T-t)\right]^2 / 2\sigma^2(T-t)}, \end{aligned}$$

where r, D_0, and σ are constants. Show

$$\frac{G(B_l^2/S', T; S, t)}{G(S', T; B_l^2/S, t)} = \frac{S'^2}{B_l^2} \left(\frac{B_l^2}{S'S}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2}.$$

6. As we know, if $\mathbf{P}$ is a symmetric matrix and all its eigenvalues are positive, then we can find a matrix $\mathbf{Q}$ satisfying the conditions $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and a positive diagonal matrix $\mathbf{\Lambda}$, so that $\mathbf{P} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$.

   (a) Show that we can require $\det \mathbf{Q} = \det \mathbf{Q}^T = 1$ without losing generality.

   (b) Set $\mathbf{R} = \mathbf{\Lambda}^{-1/2}\mathbf{Q}^T$. Show $\det \mathbf{R} = \dfrac{1}{\sqrt{\det \mathbf{P}}}$.

   (c) $\mathbf{y}$ and $\mathbf{y}_0$ are two vectors. Set $\mathbf{x} = \mathbf{R}\mathbf{y}$, $\mathbf{x}_0 = \mathbf{R}\mathbf{y}_0$, and $\eta = \dfrac{\mathbf{y}_0 - \mathbf{y}}{\sqrt{2\tau}}$. Show

   $$\frac{(\mathbf{x}_0 - \mathbf{x})^T (\mathbf{x}_0 - \mathbf{x})}{4\tau} = \frac{\eta^T \mathbf{P}^{-1} \eta}{2}.$$

7. (a) $\mathbf{S}$ is a random vector and its covariance matrix is $\mathbf{B}$, i.e., the component on the i-th row and the j-th column of $\mathbf{B}$ is equal to $\mathbf{E}\left[(S_i - \mathbf{E}[S_i])(S_j - \mathbf{E}[S_j])\right]$, S_i being the i-th component of $\mathbf{S}$. Let $\bar{\mathbf{S}} = \mathbf{A}\mathbf{S}$, $\mathbf{A}$ being a constant matrix, and its covariance matrix be $\mathbf{C}$. Find the relation among $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

   (b) How do we choose $\mathbf{A}$ so that $\mathbf{C}$ will be a diagonal matrix?