MATH 6202/8202 **Test I** **Fall 2008**

Name :______________________

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Show the details of your work !!

1. Let $F = e^{(r-D_0)(T-t)}S$, which is called the forward/futures price, and $f = Se^{-D_0(T-t)} - Ke^{-r(T-t)} = e^{-r(T-t)}(F-K)$, which is the value of a forward/futures contract. Here K is a constant and we assume that the interest rate r and the dividend yield D_0 are also constant. Suppose that F satisfies
$$dF = \bar{\mu}F dt + \sigma F dX.$$
Consider an option on a forward/futures and let the price of such an option be $V(F,t)$. Derive the PDE for V by using Itô's lemma. (Hint: Set $\Pi = V(F,t) - \Delta f = V(F,t) - \Delta e^{-r(T-t)}(F-K)$. Also please notice that a holder of a forward/futures contract will not get any thing when a stock pays dividends).

2. As we know,
$$\int_c^\infty S^n G(S)dS = a^n e^{(n^2-n)b^2/2} N\left(-\frac{\ln(c/a)+b^2/2}{b} + nb\right),$$
where
$$\begin{aligned} G(S) &= \frac{1}{\sqrt{2\pi}bS} e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2}, \\ N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi \end{aligned}$$
and a and b are positive numbers. We also know that the solution of the final value problem
$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad 0 \le t \le T, \\ V(S,T) = V_T(S), & 0 \le S \end{cases}$$
is
$$V(S,t) = e^{-r(T-t)} \int_0^\infty V_T(S')G(S',T;S,t)dS'.$$
Here
$$G(S',T;S,t) = \frac{1}{\sqrt{2\pi}bS'} e^{-\left[\ln(S'/a)+b^2/2\right]^2/2b^2},$$
where
$$a = Se^{(r-D_0)(T-t)} \quad \text{and} \quad b = \sigma\sqrt{T-t}.$$

Using these results, show that the price of a European call option is given by

$$c(S,t) = Se^{-D_0(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2),$$

where

$$\begin{aligned} d_1 &= \left[\ln \frac{Se^{(r-D_0)(T-t)}}{E} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right), \\ d_2 &= \left[\ln \frac{Se^{(r-D_0)(T-t)}}{E} - \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right). \end{aligned}$$

3. (a) A European option is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L_S} V = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S, \end{cases}$$

where

$$\mathbf{L_S} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

For an American option, the constraint is that the inequality

$$V(S,t) \ge G(S,t)$$

holds for any S and t, where $G(S,T) = V_T(S)$. Derive the linear complementarity problem for the American option.

(b) Based on the result in part a), write the the corresponding linear complementarity problem for an American put option and reformulate this problem as a free-boundary problem if $r > 0$. Is there a free boundary if $r = 0$? Why?

4. The price of a one-factor convertible bond is the solution of the linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V,\ V(S,t) - nS\right) = 0, & 0 \le S,\, 0 \le t \le T, \\ V(S,T) = \max(Z, nS) \ge nS, & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S\frac{\partial}{\partial S} - r$$

and n, Z, σ, r, and D_0 are constants. Show

$$V(S,t^*) - Ze^{-r(T-t^*)} \ge V(S,t^{**}) - Ze^{-r(T-t^{**})} \quad \text{if} \quad t^* \le t^{**}.$$

(Hint: Define $\overline{V}(S,t) = V(S,t) - Ze^{-r(T-t)}$ and show $\overline{V}(S,t^*) \ge \overline{V}(S,t^{**})$ if $t^* \le t^{**}$.)

5. Suppose that r, D_0, and σ are constant. Derive the put–call symmetry relations, which are

$$\begin{cases} C(S,t;a,b) = P\left(E^2/S,t;b,a\right)S/E, \\ P(S,t;b,a) = C\left(E^2/S,t;a,b\right)S/E, \end{cases}$$

and

$$S_{cf}(t;a,b) \times S_{pf}(t;b,a) = E^2,$$

where the first and second arguments sfter the semicolon in C, P, S_{cf}, and S_{pf} are the values of the interest rate and dividend yield, respectively.

6. Derive the jump condition for options on stocks with discrete dividends and explain its financial meaning.