MATH 6202/8202 **Test I Part A** **Fall 2009**

Name :________________________________

ID :________________________________

Show the details of your work !!

1. (a) Suppose that S satisfies

$$dS = \mu S dt + \sigma S dX.$$

Define $\xi_{10} = \dfrac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} = \dfrac{Se^{(r-D_0)(T-t)}}{E}$ and $\xi_{01} = \dfrac{Ee^{-r(T-t)}}{Se^{-D_0(T-t)}} = \dfrac{E}{Se^{(r-D_0)(T-t)}}$, where E, D_0, r, are a constant. **Show**

$$d\xi_{10} = (\mu - r + D_0)\xi_{10} dt + \sigma \xi_{10} dX$$

and

$$d\xi_{01} = (-\mu + r - D_0 + \sigma^2)\xi_{01} dt - \sigma \xi_{01} dX.$$

(b) **Derive** the Black-Scholes formula for a European put option, assuming that we know the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), \quad 0 \le S \end{cases}$$

is

$$V(S,t) = e^{-r(T-t)} \int_0^\infty V_T(S') \cdot \frac{1}{S'\sqrt{2\pi}b} e^{-\left[\ln(S'/a) + b^2/2\right]^2/2b^2} dS',$$

where $a = Se^{(r-D_0)(T-t)}$ and $b = \sigma\sqrt{T-t}$.

2. Assume

$$u(x,\tau;\xi) = \tau^{-1/2} U\left(\frac{x-\xi}{\sqrt{\tau}}\right),$$

where ξ is a parameter and U is an unknown function. **Find** such a function U that

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \int_{-\infty}^{\infty} u(x,\tau;\xi) d\xi = 1$$

hold.

3. As we know, the prices of European call and put options are solutions of the problem

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c}{\partial S^2} + (r - D_0) S \dfrac{\partial c}{\partial S} - rc = 0, & 0 \le S, \quad t \le T, \\ c(S,T) = \max(S - E, 0), & 0 \le S, \end{cases}$$

and the problem

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p}{\partial S^2} + (r - D_0) S \dfrac{\partial p}{\partial S} - rp = 0, & 0 \le S, \quad t \le T, \\ p(S,T) = \max(E - S, 0), & 0 \le S, \end{cases}$$

respectively. Let $S_0^* = Ee^{-r(T-t)}$, $S_1^* = Se^{-D_0(T-t)}$, $\xi_{10} = S_1^*/S_0^*$, and $\xi_{01} = S_0^*/S_1^*$. Define $V_0(\xi_{10}, t) = c(S,t)/S_0^*$ and $V_1(\xi_{01}, t) = p(S,t)/S_1^*$. **Find** the PDEs and final conditions for $V_0(\xi_{10}, t)$ and $V_1(\xi_{01}, t)$ and **show** that when S_1^* is replaced by S_0^* and S_0^* by S_1^* at the same time, the expression for $c(S,t)$ becomes the expression for $p(S,t)$.

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4. The American put option is the solution of the following linear complementarity problem on a finite domain:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\, \overline{V}(\xi,0) - \max(1-2\xi,0)\right) = 0, & 0 \le \xi \le 1, 0 \le \tau, \\ \overline{V}(\xi,0) = \max(1-2\xi,0), & 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\sigma^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi)+D_0\xi].$$

<u>Show</u> that if $r > 0$, then there is a free boundary at $\tau = 0$ and **<u>find</u>** the location of the free boundary at $\tau = 0$.

5. Suppose that $V(S,t)$ is the solution of the following PDE:

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V + d(S,t)\delta(t-t_i) = 0.$$

<u>Find</u> the relation between $V(S,t_i^+)$ and $V(S,t_i^-)$, and **<u>describe</u>** the financial meaning of this relation.

6. Suppose that S is the price of a dividend-paying stock and satisfies

$$dS = \mu(S,t)Sdt + \sigma S dX_1, \quad 0 \le S < \infty,$$

where dX_1 is a Wiener process and σ is another random variable. Let the dividend paid during the time period $[t, t+dt]$ be $D(S,t)dt$. Assume that for σ, the stochastic equation

$$d\sigma = p(\sigma,t)dt + q(\sigma,t)dX_2, \quad \sigma_l \le \sigma \le \sigma_u$$

holds. Here dX_2 is another Wiener process correlated with dX_1, and the correlation coefficient between them is ρdt. For options on such a stock, **<u>derive directly</u>** the PDE that contains only the unknown market price of volatility risk. Here "Directly" means "without using the general PDE for derivatives. (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where V_1 and V_2 are two different options.)