MATH 6202/8202 **Test I** **Spring 2009**

Name :____________________

ID :____________________

Show the details of your work !!

1. Suppose that S_1 and S_2 are two lognormal random variables satisfying the following stochastic differential equations:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \quad i = 1, 2,$$

where $\mu_i, \sigma_i, i = 1, 2$, are constants and $dX_i, i = 1, 2$, are two Wiener processes, i.e., $dX_i = \phi_i \sqrt{dt}$, ϕ_i being distinct standardized normal random variables, $i = 1, 2$. ϕ_1 and ϕ_2 could be correlated and

$$\mathrm{E}[\phi_i \phi_j] = \rho_{ij}, \quad i, j = 1, 2,$$

where $-1 \le \rho_{ij} \le 1$. Let $V = V(S_1, S_2, t)$. Among dV and dS_i, $i = 1, 2$, there is the following Itô's lemma:

$$dV = f dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2,$$

where

$$f = \frac{\partial V}{\partial t} + \frac{1}{2}\left(\frac{\partial^2 V}{\partial S_1^2}\sigma_1^2 S_1^2 + 2\frac{\partial^2 V}{\partial S_1 \partial S_2}\sigma_1\sigma_2 S_1 S_2 \rho_{12} + \frac{\partial^2 V}{\partial S_2^2}\sigma_2^2 S_2^2 \right).$$

Define

$$\xi_{12} = \frac{S_1}{S_2}.$$

<u>**Show**</u> that ξ_{12} satisfies the following stochastic differential equation:

$$d\xi_{12} = (\mu_1 - \mu_2 + \sigma_2^2 - \rho_{12}\sigma_1\sigma_2)\xi_{12} dt + \sigma_{12}\xi_{12} dX_{12},$$

where

$$\sigma_{12} = \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}, \quad \text{and} \quad dX_{12} = \frac{\sigma_1 dX_1 - \sigma_2 dX_2}{\sigma_{12}},$$

and that

$$\mathrm{E}[dX_{12}] = 0 \quad \text{and} \quad \mathrm{Var}[dX_{12}] = dt.$$

2. Suppose that S is a random variable that is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi} bS} e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2},$$

a and b being positive numbers. **<u>Show</u>**

$$\int_0^c S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(\frac{\ln(c/a)+b^2/2}{b} - nb\right),$$

and

$$\begin{aligned}\int_0^c \ln S\, G(S)\, dS &= \frac{-b}{\sqrt{2\pi}} e^{-\left[\ln(c/a)+b^2/2\right]^2/2b^2} \\ &\quad + \left(\ln a - b^2/2\right) N\left(\frac{\ln(c/a)+b^2/2}{b}\right),\end{aligned}$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2} d\xi.$$

3. **<u>Derive</u>** the Black-Scholes formula for a European call option, assuming that you have proved the following results:.

   (a) let

   $$x = \ln S + (r - D_0 - \sigma^2/2)(T-t), \quad \bar{\tau} = \sigma^2(T-t)/2$$

   and

   $$V(S,t) = e^{-r(T-t)} u(x, \bar{\tau}),$$

   then the problem

   $$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), \quad 0 \le S \end{cases}$$

   can be converted into the following problem

   $$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x,0) = V_T(e^x), & -\infty < x < \infty; \end{cases}$$

(b) for the following problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x, 0) = u_0(x), & -\infty < x < \infty, \end{cases}$$

the solution is

$$u(x, \bar{\tau}) = \frac{1}{2\sqrt{\pi\bar{\tau}}} \int_{-\infty}^{\infty} u_0(\xi) e^{-(\xi - x)^2/4\bar{\tau}} d\xi.$$

4. (a) **Show** that if $D_0 = 0$, then $c(S, t) \ge \max(S - E, 0)$, which means that for this case the value of an American call option is the same as the value of a European call option.

   (b) **Show** that if $r = 0$, then $p(S, t) \ge \max(E - S, 0)$, which means that for this case the value of an American put option is the same as the value of a European put option.

5. The American put option is the solution of the following linear complementarity problem on a finite domain:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V}, \, \overline{V}(\xi, 0) - \max(1 - 2\xi, 0)\right) = 0, & 0 \le \xi \le 1, 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(1 - 2\xi, 0), & 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\sigma^2(\xi)\xi^2(1 - \xi)^2 \frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1 - \xi)\frac{\partial}{\partial \xi} - [r(1 - \xi) + D_0\xi].$$

Show that if $r > 0$, then there is a free boundary at $\tau = 0$ and **find** the location of the free boundary at $\tau = 0$.

6. Suppose that r, D_0, and σ are constant. **Derive** the put–call symmetry relations.