MATH 6202/8202 **Test I (Part 1)** **Spring 2010**

Name :____________________

ID :____________________

Show the details of your work !!

1. Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0\ ,\ 0 \le S,\ \ t \le T, \\ V(S,T) = V_T(S),\ \ 0 \le S. \end{cases}$$

Let $\xi = \dfrac{S}{S + P_m}$, $\tau = T - t$ and $V(S,t) = (S + P_m)\overline{V}(\xi, \tau)$, where P_m is a positive constant. Show that $\overline{V}(\xi, \tau)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad\quad -[r(1-\xi) + D_0\xi]\overline{V}, \qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = \dfrac{(1-\xi)}{P_m}V_T\left(\dfrac{P_m\xi}{1-\xi}\right), \end{cases}$$

where $\bar{\sigma}(\xi) = \sigma\left(\dfrac{P_m\xi}{1-\xi}\right)$.

2. (a) Suppose that S is a random variable which is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi}bS}e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2},$$

a and b being positive numbers. Show that for any real number n

$$\int_c^\infty S^n G(S)dS = a^n e^{(n^2-n)b^2/2}N\left(-\frac{\ln(c/a) + b^2/2}{b} + nb\right);$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^z e^{-\xi^2/2}d\xi.$$

(b) Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - r B_c = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), \qquad\qquad\qquad 0 \le S, \end{cases}$$

where σ, r, D_0, Z, and n are constants. Show that if $D_0 \le 0$, then

$$B_c(S,t) \ge \max\left(Z e^{-r(T-t)}, nS\right) \quad \text{for} \quad 0 \le t \le T.$$

3. As we know, when the LC problem of an American call option is formulated as a free-boundary problem, on the free boundary $S = S_f(t) \ge \max(E, rE/D_0)$, we need to require $C\left(S_f(t), t\right) = \max\left(S_f(t) - E, 0\right) = S_f(t) - E$ and $\dfrac{\partial C\left(S_f(t), t\right)}{\partial S} = 1$, where $C\left(S, t\right)$ and $\max\left(S - E, 0\right)$ are the solution of the free-boundary problem and the constraint. Show that if $C(S,t) \ge 0$ and $\dfrac{\partial C^2\left(S, t\right)}{\partial S^2} \ge 0$ for $S < S_f(t)$, then the solution of the free-bounary problem satisfies the LC condition

$$\min\left(-\frac{\partial C}{\partial t} - \mathbf{L}_S C,\ C - \max(S - E, 0)\right) = 0,$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r.$$

MATH 6202/8202 Test I (Part 2) Spring 2010

Name :________________

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Show the details of your work !!

4. Suppose that S_1, S_2, $\cdots$, S_n are n lognormal random variables satisfying the following stochastic differential equations:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \quad i = 1, 2, \cdots, n,$$

where $\mu_i, \sigma_i, i = 1, 2, \cdots, n$, are constants and $dX_i, i = 1, 2, \cdots, n$, are n Wiener processes, i.e., $dX_i = \phi_i \sqrt{dt}$, ϕ_i being distinct standardized normal random variables, $i = 1, 2, \cdots, n$. ϕ_i and ϕ_j could be correlated and

$$\mathrm{E}[\phi_i \phi_j] = \rho_{ij}, \quad i, j = 1, 2, \cdots, n,$$

where $-1 \leq \rho_{ij} \leq 1$. Define

$$\xi_{ij} = \frac{S_i}{S_j}, \quad i \neq j.$$

(a) Show that ξ_{ij} satisfies the following stochastic differential equation

$$d\xi_{ij} = (\mu_i - \mu_j + \sigma_j^2 - \rho_{ij}\sigma_i\sigma_j)\xi_{ij}dt + \sigma_{ij}\xi_{ij}dX_{ij},$$

where

$$\sigma_{ij} = \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2}$$

and dX_{ij} is a Wiener process defined by

$$dX_{ij} = \frac{\sigma_i dX_i - \sigma_j dX_j}{\sigma_{ij}}.$$

That is, $\xi_{ij} = S_i/S_j$ is also a lognormal variable and its volatility is σ_{ij}.

(b) Define

$$\rho_{ijk} = \frac{\rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k + \sigma_k^2}{\sigma_{ik}\sigma_{jk}}.$$

Show

$$\mathrm{E}[dX_{ik}dX_{jk}] = \rho_{ijk}dt,$$

i.e., ρ_{ijk} is the correlation coefficient between the Wiener processes related to ξ_{ik} and ξ_{jk}.

5. (a) The price of a one-factor convertible bond paying no coupon is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V,\ V(S,t) - nS\right) = 0, & 0 \le S,\ 0 \le t \le T, \\ V(S,T) = \max(Z, nS) \ge nS, & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S\frac{\partial}{\partial S} - r$$

and n, Z, σ, r, and D_0 are positive constants. Show

$$V(S,t^*) - Ze^{-r(T-t^*)} \ge V(S,t^{**}) - Ze^{-r(T-t^{**})} \quad \text{if} \quad t^* \le t^{**}.$$

(b) Can you prove that $V(S,t^*) \ge V(S,t^{**})$ for $t^* \le t^{**}$ by using the method used in a)? If your answer is "Yes", give a proof; otherwise explain why you cannot.

6. Consider a two-factor convertible bond paying coupons with a rate k. For such a convertible bond, derive directly the partial differential equation that contains only the unknown market price of risk for the spot interest rate. "Directly" means "without using the general PDE for derivatives". (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where V_1 and V_2 are two different convertible bonds.)