MATH 6202/8202 **Test I (Part 1)** **Spring 2011**

Name :____________________
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Show the details of your work !!

1. (a) Suppose that S is a random variable which is defined on $[0,\infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi}bS} e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2},$$

a and b being positive numbers. **Show** that for any real number n

$$\int_0^c S^n G(S)dS = a^n e^{(n^2-n)b^2/2} N\left(\frac{\ln(c/a)+b^2/2}{b} - nb\right),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2} d\xi.$$

(b) Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r-D_0)S\dfrac{\partial B_c}{\partial S} - rB_c = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), \qquad\qquad 0 \le S, \end{cases}$$

where σ, r, D_0, Z, and n are constants. **Show** that if $D_0 \le 0$, then

$$B_c(S,t) \ge \max\left(Ze^{-r(T-t)}, nS\right) \quad \text{for} \quad 0 \le t \le T.$$

2. Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r-D_0)S\dfrac{\partial V}{\partial S} - rV = 0\ ,\ 0 \le S, \ \ t \le T, \\ V(S,T) = V_T(S), \ \ 0 \le S. \end{cases}$$

Let $\xi = \dfrac{S}{S+P_m}$, $\tau = T - t$, and $V(S,t) = (S+P_m)\overline{V}(\xi,\tau)$, where P_m is a positive constant. **Show** that $\overline{V}(\xi,\tau)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} & \\ \qquad -[r(1-\xi) + D_0\xi]\overline{V}, & 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi,0) = \dfrac{1-\xi}{P_m}V_T\left(\dfrac{P_m\xi}{1-\xi}\right), & 0 \le \xi \le 1, \end{cases}$$

where

$$\bar{\sigma}(\xi) = \sigma\left(\frac{P_m\xi}{1-\xi}\right).$$

3. (a) **Explain** why an American option is always worth at least as much as its intrinsic value.

   (b) Suppose that for an American option, the constraint is $G(S,t)$, its value at time t is $V(S,t)$, and $V(S,t) = G(S,t)$ on (A,B). Assume that when $V(S,t)$ is given as the value of a European option at t, the value of the European option at $t - \Delta t$ for a positive and very small Δt is $v(S, t-\Delta t)$. **Explain** that if in an open interval containing $S^* \in (A,B)$, $v(S,t-\Delta t) < G(S,t-\Delta t)$, then for the American option a fair value at the point $(S^*, t-\Delta t)$ should be $G(S^*, t-\Delta t)$.

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Show the details of your work !!

4. The price of a one-factor convertible bond is the solution of the linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V - kZ,\ V(S,t) - nS\right) = 0, & 0 \le S,\ 0 \le t \le T, \\ V(S,T) = \max(Z, nS) \ge nS, & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r,$$

and k, Z, n, σ, r, and D_0 are constants. **Show** that if $D_0 > 0$, then the solution of a one-factor convertible bond must involve a free boundary and its location at $t = T$ is $S = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right)$. Also, **derive** the corresponding free-boundary problem if this problem has only one free boundary.

5. The solution of the following free-boundary problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 P_\infty}{dS^2} + (r - D_0) S \dfrac{dP_\infty}{dS} - rP_\infty = 0, & S_f \le S, \\ P_\infty(S_f) = E - S_f, & \\ \dfrac{dP_\infty(S_f)}{dS} = -1 & \end{cases}$$

is $P_\infty(S) = \frac{-S_f}{\alpha_-}\left(\frac{S}{S_f}\right)^{\alpha_-}$ for $S_f \le S$, where

$$\alpha_- = \frac{1}{\sigma^2}\left[-\left(r - D_0 - \frac{1}{2}\sigma^2\right) - \sqrt{\left(r - D_0 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}\right]$$

and $S_f = \dfrac{E}{1 - 1/\alpha_-}$.

(a) Suppose $f(b) = f'(b) = 0$ and $a < b$. **Show** that if $f''(x) \geq 0$ for $x \in [a, b]$, then $f(x) \geq 0$ for $x \in [a, b]$.

(b) Define

$$P_\infty(S) = \begin{cases} E - S, & 0 \leq S < S_f, \\ \dfrac{-S_f}{\alpha_-}\left(\dfrac{S}{S_f}\right)^{\alpha_-}, & S_f \leq S. \end{cases}$$

By substituting this expression of P_∞ into

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{d^2 P_\infty}{dS^2} + (r - D_0) S \frac{dP_\infty}{dS} - rP_\infty\right], \ P_\infty - \max(E - S, 0)\right),$$

show that it is always equal to zero, that is, $P_\infty(S)$ is a solution of the perpetual American put option.

6. Suppose that ξ_1 anf ξ_2 satisfy the system of stochastic differential equations:

$$d\xi_i = \mu_i(\xi_1, \xi_2, t)dt + \sigma_i(\xi_1, \xi_2, t)dX_i, \quad i = 1, 2,$$

where dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij} dt$ with $-1 \leq \rho_{ij} \leq 1$. Define

$$\begin{cases} Z_1(\xi_1) &= Z_{1,l} + \xi_1 (1 - Z_{1,l}), \\ Z_2(\xi_1, \xi_2) &= Z_{2,l} + \xi_2 \left[Z_1(\xi_1) - Z_{2,l}\right] \\ &= Z_{2,l} + \xi_2 \left[Z_{1,l} + \xi_1 (1 - Z_{1,l}) - Z_{2,l}\right]. \end{cases}$$

Assume that $Z_1(\xi_1)$ and $Z_2(\xi_1, \xi_2)$ represent prices of two securities. Let $V(\xi_1, \xi_2, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2)$ and using Itô's lemma, **show** that $V(\xi_1, \xi_2, t)$ satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{rZ_1}{1 - Z_{1,l}} \frac{\partial V}{\partial \xi_1}$$
$$+ \left[\frac{r(Z_2 - Z_1 \xi_2)}{Z_1 - Z_{2,l}} - \frac{\sigma_1 \sigma_2 \rho_{1,2}(1 - Z_{1,l})}{Z_1 - Z_{2,l}}\right] \frac{\partial V}{\partial \xi_2} - rV = 0.$$