MATH 6202/8202 **Test I (Part 1)** **Spring 2012**

Name :____________________
ID :____________________

Show the details of your work !!

1. Suppose that S is a random variable which is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi} bS} e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2},$$

a and b being positive numbers. **Show**

(a) for any real number n

$$\int_c^\infty S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(-\frac{\ln(c/a)+b^2/2}{b} + nb\right),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi;$$

(b)

$$\int_0^c \ln S\, G(S)\, dS$$
$$= \frac{-b}{\sqrt{2\pi}} e^{-\left[\ln(c/a)+b^2/2\right]^2/2b^2} + \left(\ln a - b^2/2\right) N\left(\frac{\ln(c/a)+b^2/2}{b}\right).$$

2. (a) Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - r B_c = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), \qquad\qquad 0 \le S, \end{cases}$$

where σ, r, D_0, Z, and n are constants. **Show** that if $D_0 \le 0$, then

$$B_c(S,t) \ge \max\left(Ze^{-r(T-t)}, nS\right) \quad \text{for} \quad 0 \le t \le T.$$

(b) Suppose that $V(S,t)$ is the solution of the following PDE:

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V + d(S,t)\delta(t-t_i) = 0.$$

Find the relation between $V(S,t_i^+)$ and $V(S,t_i^-)$, and **describe** the financial meaning of this relation.

3. (a) Suppose $\sigma = \sigma(S,t)$, $r = r(t)$ and $D_0 = D_0(S,t)$. **Show** that the problem of pricing a put option can always be converted into a problem of pricing a call option.

(b) Let the exercise price be E. Suppose that r, D_0 are constants and $\sigma = \sigma(S)$. Based the result in (a), **Show**

$$\begin{aligned} P(S,t;b,\ a,\ \sigma(S)) &= C\left(E^2/S,t;a,\ b,\ \sigma(S)\right)S/E, \\ C\left(S,t;a,b,\sigma(S)\right) &= P\left(E^2/S,t;b,a,\sigma(S)\right)S/E \end{aligned}$$

and

$$S_{cf}(t;a,\ b,\ \sigma(S)) \cdot S_{pf}(t;b,\ a,\ \sigma(E^2/S)) = E^2.$$

Here the first, second and third parameters after the semicolon in P, C, S_{pf} and S_{cf} are the interest rate, the dividend yield and the volatility function, respectively.

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4. Suppose that S is the price of a dividend-paying stock and satisfies

$$dS = \mu(S,t)Sdt + \sigma S dX_1, \quad 0 \le S < \infty,$$

where dX_1 is a Wiener process and σ is another random variable. Let the dividend paid during the time period $[t, t+dt]$ be $D(S,t)dt$. Assume that for σ, the stochastic equation

$$d\sigma = p(\sigma,t)dt + q(\sigma,t)dX_2, \quad \sigma_l \le \sigma \le \sigma_u$$

holds. Here, $p(\sigma,t)$ and $q(\sigma,t)$ are differentiable functions and satisfy the reversion conditions. dX_2 is another Wiener process correlated with dX_1, and the correlation coefficient between them is ρdt. For options on such a stock, **derive** directly the partial differential equation that contains only the unknown market price of risk for the volatility. Here "Directly" means "without using the general PDE for derivatives." (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where V_1 and V_2 are two different options.)

5. As we know, when the LC problem of an American call option is formulated as a free-boundary problem, on the free boundary $S = S_f(t) \ge \max(E, rE/D_0)$, we need to require $C\left(S_f(t), t\right) = \max\left(S_f(t) - E, 0\right) = S_f(t) - E$ and $\dfrac{\partial C\left(S_f(t), t\right)}{\partial S} = 1$, where $C\left(S, t\right)$ and $\max\left(S - E, 0\right)$ are the solution of the free-boundary problem and the constraint. **Show** that if $C(S,t) \ge 0$ and $\dfrac{\partial C^2\left(S,t\right)}{\partial S^2} \ge 0$ for $S < S_f(t)$, then the solution of the free-bounary problem satisfies the LC condition

$$\min\left(-\frac{\partial C}{\partial t} - \mathbf{L}_S C,\ C - \max(S-E,0)\right) = 0,$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r,$$

that is, for $S \in [0, S_f(t))$, $C(S,t)$ truely is a solution of the LC problem.

6. (a) The price of a one-factor convertible bond paying no coupon is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V,\ V(S,t) - nS\right) = 0, & 0 \le S,\ 0 \le t \le T, \\ V(S,T) = \max(Z, nS) \ge nS, & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S \frac{\partial}{\partial S} - r$$

and n, Z, σ, r, and D_0 are positive constants. **Show**

$$V(S,t^*) - Ze^{-r(T-t^*)} \ge V(S,t^{**}) - Ze^{-r(T-t^{**})} \quad \text{if} \quad t^* \le t^{**}.$$

(Hint: Define $\overline{V}(S,t) = V(S,t) - Ze^{-r(T-t)}$ and show $\overline{V}(S,t^*) \ge \overline{V}(S,t^{**})$ if $t^* \le t^{**}$.)

(b) Can you prove that $V(S,t^*) \ge V(S,t^{**})$ for $t^* \le t^{**}$ by using the method used in (a)? **If your answer is "Yes", give a proof; otherwise explain why you cannot**.

(c) "A holder of a convertible bond at time t^* has "more rights" than a holder of a convertible bond at time t^{**} does if $t^* \le t^{**}$, so the premium at t^* should be higher than the premium at t^{**}, i.e., the inequality $V(S,t^*) \ge V(S,t^{**})$ should hold for any $t^* \le t^{**}$." **Do you think that this statement is true and why**?