MATH 6202 **Test II** **Fall 2008**

Name :________________________________

ID :________________________________

Show the details of your work !!

1. Consider an average strike option with discrete arithmetic averaging. Assume that the stock pays dividends and that during the time step $[t, t+dt]$, the dividend payment is $D(S,t)dt$. Take S and
$$I = \frac{1}{K}\int_0^t S(\tau)f(\tau)d\tau$$
as state variables, where
$$f(t) = \sum_{i=1}^{K}\delta(t-t_i).$$
In this case the PDE for options is
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (rS - D)\frac{\partial V}{\partial S} + \frac{Sf(t)}{K}\frac{\partial V}{\partial I} - rV = 0.$$
   (a) Find the jump condition at $t = t_i$, $i = 1, 2, \cdots, K$.

   (b) Under the assumption $D(S,t) = D_0 S$ and $V(S,I,T) = \max(\pm(\alpha S - I), 0)$, reduce an average strike option problem to a problem with only two independent variables and the payoff to a function with only one independent variable.

2. (a) Find the solution of the problem:
$$\begin{cases} \dfrac{d\varphi_1(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2}\dfrac{d\varphi_2(1/\eta)}{d\eta}, & \text{for } \eta\in[0,1], \\ \varphi_1(1) = \varphi_2(1), \end{cases}$$
where $\varphi_2(\eta) = \max(\eta-\beta, 0)$ defined on $\eta\in[1,\infty)$. Here $\beta > 1$ and $r \neq D_0$.

   (b) Show
$$\begin{aligned} &\int_0^1 \varphi_1(\eta')\, G(\eta', T; \eta, t)\, d\eta' + \int_1^\infty \max(\eta'-\beta, 0)\, G(\eta', T; \eta, t)\, d\eta' \\ = \;& \frac{\sigma^2\beta^{-2(r-D_0)/\sigma^2}}{2(r-D_0)} N\left(\frac{-\ln(\beta\eta) + (\mu+\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ & -\frac{\sigma^2}{2(r-D_0)}\eta^{2(r-D_0)/\sigma^2} e^{-(r-D_0)\tau} N\left(\frac{-\ln(\beta\eta) - \mu\tau}{\sigma\sqrt{\tau}}\right) \\ & +\eta e^{(D_0-r)\tau} N\left(\frac{\ln(\eta/\beta) - \mu\tau}{\sigma\sqrt{\tau}}\right) \\ & -\beta N\left(\frac{\ln(\eta/\beta) - (\mu+\sigma^2)\tau}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

where

$$G\left(\eta', T; \eta, t\right) = \frac{1}{\sigma\sqrt{2\pi\left(T-t\right)}\eta'} e^{-\left[\ln(\eta'/\eta) - \left(D_0 - r - \sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}$$

and $\mu = r - D_0 - \sigma^2/2$.

3. Consider the following problem

$$\begin{cases} \dfrac{\partial \overline{V}_1(\mathbf{y},\tau)}{\partial \tau} = \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \overline{V}_1(\mathbf{y},\tau)}{\partial y_i \partial y_j}, & -\infty < \mathbf{y} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}(\mathbf{y}, 0) = V_{1T}(\mathbf{y}), & -\infty < \mathbf{y} < \infty, \end{cases}$$

where

$$V_{1T}(\mathbf{y}) \equiv V_T\left(e^{\sigma_1 y_1/\sqrt{2}}, e^{\sigma_2 y_2/\sqrt{2}}, \cdots, e^{\sigma_n y_n/\sqrt{2}}\right).$$

(a) Let

$$\mathbf{x} = \mathbf{R}\mathbf{y}$$

and

$$\overline{V}_2(\mathbf{x},\tau) = \overline{V}_1(\mathbf{y},\tau),$$

where $\mathbf{R}$ is a contant matrix:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}.$$

Find the equation and initial condition for $\overline{V}_2(\mathbf{x},\tau)$.

(b) Find $\mathbf{R}$ such that $\overline{V}_2(\mathbf{x},\tau)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{V}_2(\mathbf{x},\tau)}{\partial \tau} = \displaystyle\sum_{l=1}^{n} \frac{\partial^2 \overline{V}_2(\mathbf{x},\tau)}{\partial x_l^2}, & -\infty < \mathbf{x} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}_2(\mathbf{x}, 0) = V_{2T}(\mathbf{x}), & -\infty < \mathbf{x} < \infty, \end{cases}$$

where $V_{2T}(\mathbf{x}) \equiv V_{1T}(\mathbf{R}^{-1}\mathbf{x})$.

4. Consider the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\alpha r \dfrac{\partial^2 V}{\partial r^2} + (\mu - \gamma r)\dfrac{\partial V}{\partial r} - rV = 0, & 0 \le r, \quad 0 \le t \le T, \\ V(r, T; T) = Z, & 0 \le r, \end{cases}$$

where α, μ, γ and Z are constants.

(a) Show that this problem has a solution in the form

$$V(r,t;T) = Ze^{A(t,T)-rB(t,T)}$$

and A and B are the solution of the system of ordinary differential equations

$$\begin{cases} \dfrac{dA}{dt} = \mu B, \\ \dfrac{dB}{dt} = \dfrac{1}{2}\alpha B^2 + \gamma B - 1 \end{cases}$$

with the conditions $A(T,T) = 0$ and $B(T,T) = 0$;

(b) Suppose that we already find the function B:

$$B = \frac{2(e^{\psi(T-t)} - 1)}{(\gamma+\psi)e^{\psi(T-t)} - (\gamma-\psi)},$$

where $\psi = \sqrt{\gamma^2 + 2\alpha}$. By solving the first ODE, show

$$A = \ln\left(\frac{2\psi e^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)e^{\psi(T-t)} - (\gamma-\psi)}\right)^{2\mu/\alpha}.$$

(This problem is related to the Cox-Ingersoll-Ross model of spot interest rate.)

5. (a) Suppose that there is a domain Ω on the (Z_1, Z_2)-plane, the boundary of Ω is Γ, and $(n_1, n_2)^T$ is the outer normal vector of the boundary Γ. Assume that Z_1 and Z_2 are two stochastic processes and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i(Z_1, Z_2, t)dt + \sigma_i(Z_1, Z_2, t)dX_i \quad \text{with} \quad \sigma_i \geq 0, \quad i = 1, 2,$$

where $dX_i, i = 1, 2$, are the Wiener processes and $\mathrm{E}\left[dX_1 dX_2\right] = \rho_{12}dt$ with $\rho_{12} \in [-1, 1]$. Suppose that at $t = 0$, $(Z_1, Z_2) \in \Omega$. Show that in order to guarantee $(Z_1, Z_2) \in \Omega$ for any time $t \in [0, T]$, we need to require, for any $t \in [0, T]$ and for any point on Γ, the following condition to be held:

i. if $n_1 \neq 0$ and $n_2 = 0$, then

$$\begin{cases} n_1\mu_1 \leq 0, \\ \sigma_1 = 0; \end{cases}$$

ii. if $n_1 = 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_2\mu_2 \leq 0, \\ \sigma_2 = 0; \end{cases}$$

iii. if $n_1 \neq 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_1\mu_1 + n_2\mu_2 \leq 0, \\ n_1\sigma_1 - \text{sign}(n_1 n_2) n_2 \sigma_2 = 0, \quad \text{and} \quad \rho_{12} = -\text{sign}(n_1 n_2), \end{cases}$$

where

$$\text{sign}(n_1 n_2) = \begin{cases} 1, & \text{if } n_1 n_2 > 0, \\ -1, & \text{if } n_1 n_2 < 0. \end{cases}$$

If a point is a corner point, then there are two normals and we need to require this condition to be held for the two outer normal vectors.

(b) Suppose that the domain Ω is $Z_{1l} \leq Z_1 \leq 1$ and $Z_{2l} \leq Z_2 \leq Z_1$, where Z_{1l} and Z_{2l} are constants, and $Z_{1l} \geq Z_{2l}$. Find the concrete condition corresponding to the condition given in a) on each segment of the boundary.

6. Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq S, \quad 0 \leq t \leq T, \\ B_c(S, T) = \max(Z, nS), \qquad\qquad\qquad 0 \leq S, \end{cases}$$

where σ, r, D_0, k, Z, and n are constants. Show that if $D_0 \leq 0$, then

$$B_c(S, t) \geq nS \quad \text{for} \quad 0 \leq t \leq T.$$

(Hint: Define $\overline{B}_c(S, t) = B_c(S, t) - b_0(t)$, where $b_0(t) = \dfrac{kZ}{r}\left(1 - e^{-r(T-t)}\right)$. First show that $\overline{B}_c(S, t)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{B}_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{B}_c}{\partial S^2} + (r - D_0) S \dfrac{\partial \overline{B}_c}{\partial S} - r\overline{B}_c = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq S, \quad 0 \leq t \leq T, \\ \overline{B}_c(S, T) = \max(Z, nS), \qquad\qquad\qquad 0 \leq S. \end{cases}$$

Then show $\overline{B}_c(S, t) \geq nS$ and $b_0(t) \geq 0$ for $0 \leq t \leq T$. Finally show $B_c(S, t) \geq nS$.)
(**Remark**: If the solution of this problem fulfills the constraint condition

$$B_c(S, t) \geq nS \quad \text{for} \quad 0 \leq t \leq T,$$

then the solution of the problem above represents the price of a one-factor convertible bond. In this case, the solution of a one-factor convertible bond does not involve any free boundary. Therefore, no free boundary will be encountered when one prices a one-factor convertible bond with $D_0 \leq 0$.)