MATH 6202 **Test II (Part A)** **Fall 2009**

Name :______________________________

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Show the details of your work !!

1. (4 points) **Show** that a European up-and-out put option with $B_u > E$ plus a European up-and-in put option with the same parameters is equal to a vanilla European put option.

2. (a) (4 points) **Find** the solution of the problem:

$$\begin{cases} \dfrac{d\varphi_1(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_2(1/\eta)}{d\eta}, \qquad \text{for } \eta \in [0,1], \\ \varphi_1(1) = \varphi_2(1), \end{cases}$$

where $\varphi_2(\eta) = \max(\eta - \beta, 0)$ defined on $\eta \in [1, \infty)$. Here $\beta > 1$ and $r \neq D_0$.

(b) (4 points) **Show**

$$\begin{aligned} &\int_0^1 \varphi_1(\eta') G(\eta', T; \eta, t)\, d\eta' + \int_1^\infty \max(\eta' - \beta, 0)\, G(\eta', T; \eta, t)\, d\eta' \\ = \; & \frac{\sigma^2 \beta^{-2(r-D_0)/\sigma^2}}{2(r-D_0)} N\left(\frac{-\ln(\beta\eta) + (\mu+\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ & - \frac{\sigma^2}{2(r-D_0)} \eta^{2(r-D_0)/\sigma^2} e^{-(r-D_0)\tau} N\left(\frac{-\ln(\beta\eta) - \mu\tau}{\sigma\sqrt{\tau}}\right) \\ & + \eta e^{(D_0-r)\tau} N\left(\frac{\ln(\eta/\beta) - \mu\tau}{\sigma\sqrt{\tau}}\right) \\ & - \beta N\left(\frac{\ln(\eta/\beta) - (\mu+\sigma^2)\tau}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

where

$$G(\eta', T; \eta, t) = \frac{1}{\sigma\sqrt{2\pi(T-t)}\eta'} e^{-\left[\ln(\eta'/\eta) - \left(D_0 - r - \sigma^2/2\right)(T-t)\right]^2 / 2\sigma^2(T-t)}$$

and $\mu = r - D_0 - \sigma^2/2$.

3. (8 points) Consider the following problem of the European option on the minimum of assets S_0, S_1, and S_2:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_1^2 S_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \dfrac{1}{2}\sigma_2^2 S_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ \qquad + (r - D_{01}) S_1 \dfrac{\partial V}{\partial S_1} + (r - D_{02}) S_2 \dfrac{\partial V}{\partial S_2} - rV = 0, \\ \qquad\qquad 0 \le S_1, \quad 0 \le S_2, \quad 0 \le t \le T, \\ V(S_1, S_2, T) = \min(S_0, S_1, S_2), \qquad 0 \le S_1, \quad 0 \le S_2. \end{cases}$$

Assume that we already prove that if we set $\xi_{10} = \dfrac{S_1^*}{S_0^*} = \dfrac{S_1 e^{-D_{01}(T-t)}}{S_0 e^{-r(T-t)}}$, $\xi_{20} = \dfrac{S_2^*}{S_0^*} = \dfrac{S_2 e^{-D_{02}(T-t)}}{S_0 e^{-r(T-t)}}$, and $V_0(\xi_{10}, \xi_{20}, t) = \dfrac{V}{S_0^*} = \dfrac{V}{S_0 e^{-r(T-t)}}$, then $V_0(\xi_{10}, \xi_{20}, t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V_0}{\partial t} + \dfrac{1}{2}\sigma_{10}^2 \xi_{10}^2 \dfrac{\partial^2 V_0}{\partial \xi_{10}^2} + \rho_{120}\sigma_{10}\sigma_{20}\xi_{10}\xi_{20}\dfrac{\partial^2 V_0}{\partial \xi_{10}\partial \xi_{20}} + \dfrac{1}{2}\sigma_{20}^2\xi_{20}^2\dfrac{\partial^2 V_0}{\partial \xi_{20}^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \xi_{10}, \quad 0 \le \xi_{20}, \quad 0 \le t \le T, \\ V_0(\xi_{10}, \xi_{20}, T) = \min(1, \xi_{10}, \xi_{20}), \qquad\qquad 0 \le \xi_{10}, \quad 0 \le \xi_{20}. \end{cases}$$

Here σ_{10} and σ_{20} are the volatilities of ξ_{10} and ξ_{20}, respectively, and ρ_{120} is the correlation coefficient between the Wiener processes associated with the lognormal variables ξ_{10} and ξ_{20}. For them there are the following expressions: $\sigma_{10} = \sigma_1$, $\sigma_{20} = \sigma_2$ and $\rho_{120} = \rho_{12}$. We also assume that we already know that $V_0(\xi_{10}, \xi_{20}, t)$ has the following expression:

$$V_0(\xi_{10}, \xi_{20}, t) = \int_0^\infty \int_0^\infty \min(1, \xi_{10T}, \xi_{20T})\, \psi\, d\xi_{10T}\, d\xi_{20T},$$

Here $\psi = \psi(\xi_{10T}, \xi_{20T}; \xi_{10}, \xi_{20}, t)$ and its concrete expression is

$$\psi(\xi_{10T}, \xi_{20T}; \xi_{10}, \xi_{20}, t) = \frac{\exp\left(-\eta^T \mathbf{P}^{-1}\eta/2\right)}{2\pi\tau\sqrt{\det \mathbf{P}}\,\sigma_{10}\xi_{10T}\sigma_{20}\xi_{20T}},$$

where

$$\begin{aligned} \tau &= T - t, \\ \eta &\equiv \begin{bmatrix} \eta_1(\xi_{10T}; \xi_{10}, \tau) \\ \eta_2(\xi_{20T}; \xi_{20}, \tau) \end{bmatrix} = \begin{bmatrix} \dfrac{\ln \xi_{10T} - \ln \xi_{10} + \sigma_{10}^2\tau/2}{\sigma_{10}\sqrt{\tau}} \\ \dfrac{\ln \xi_{20T} - \ln \xi_{20} + \sigma_{20}^2\tau/2}{\sigma_{20}\sqrt{\tau}} \end{bmatrix}, \\ \mathbf{P} &= \begin{bmatrix} 1 & \rho_{120} \\ \rho_{120} & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{1-\rho_{120}^2}\begin{bmatrix} 1 & -\rho_{120} \\ -\rho_{120} & 1 \end{bmatrix}, \end{aligned}$$

and

$$\eta^T \mathbf{P}^{-1}\eta = \frac{\eta_1^2 - 2\rho_{120}\eta_1\eta_2 + \eta_2^2}{1-\rho_{120}^2}.$$

Define

$$N_2(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \exp\left(-\frac{\eta_1^2 - 2\rho\eta_1\eta_2 + \eta_2^2}{2(1-\rho^2)}\right) d\eta_1 d\eta_2.$$

Based the results given above, **express** the solution of the European option on the minimum of assets S_0, S_1, and S_2 in term of the function $N_2(x_1, x_2; \rho)$. (If the rule "$0 \to 1$, $1 \to 2$, and $2 \to 0$" is used, you need to give a proof before using it.)

MATH 6202 **Test II (Part B)** **Fall 2009**

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Show the details of your work !!

4. (a) (4 points) Suppose the spot interest rate is a known function $r(t)$. Consider a bond with a face value Z and assume that it pays a coupon with a coupon rate $k(t)$, that is, during a time interval $(t, t+dt]$, the coupon payment is $Zk(t)dt$. **Show** that the value of the bond is

$$V(t) = Ze^{-\int_t^T r(\tau)d\tau}\left[1 + \int_t^T k(\bar{\tau})e^{\int_{\bar{\tau}}^T r(\tau)d\tau}d\bar{\tau}\right].$$

(b) (4 points) Suppose that the bond pays coupon payments at three specified dates T_1, T_2, and T_3 before the maturity date T and the payments are Zk_1, Zk_2, and Zk_3, respectively. **According to the formula given in (a)** and assuming $T_1 < T_2 < T_3$, **find** the values of the bond for $t \in [0, T_1), t \in (T_1, T_2)$, $t \in (T_2, T_3)$, and $t \in (T_3, T]$, respectively, and **give** a financial interpretation of these expressions.

5. (4 points) Let $V_{s1k}(r, T)$ denote the price of a $(k/2)$-year zero-coupon bond and $V_{s1}(r, T) = \sum_{k=1}^{2N} V_{s1k}(r, T)$. In order to obtain $V_{s1}(r, T) = \sum_{k=1}^{2N} V_{s1k}(r, T)$, we can use one of the following two methods. The first one is to solve the problem:

$$\begin{cases} \dfrac{\partial V_{s1}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{s1}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{s1}}{\partial r} - rV_{s1} \\ \qquad + \sum_{k=1}^{2N} \delta(t - T - k/2) = 0, \\ \qquad\qquad r_l \le r \le r_u, \quad T \le t \le T + N, \\ V_{s1}(r, T + N) = 0, \qquad r_l \le r \le r_u. \end{cases}$$

The second is to solve the following problems

$$\begin{cases} \dfrac{\partial V_{s1k}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{s1k}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{s1k}}{\partial r} - rV_{s1k} = 0, \qquad r_l \le r \le r_u, \\ \qquad\qquad T \le t \le T + k/2, \\ V_{s1k}(r, T + k/2) = 1, \qquad r_l \le r \le r_u, \end{cases}$$

$k = 1, 2, \cdots, 2N$, and then calculate $V_{s1}(r, T)$ by adding $V_{s1k}(r, T)$, $k = 1, 2, \cdots, 2N$, together. **Explain** which of them is better and why?

6. (a) (6 points) Suppose that there is a domain Ω on the (Z_1, Z_2)-plane, the boundary of Ω is Γ, and $(n_1, n_2)^T$ is the outer normal vector of the boundary Γ. Assume that Z_1 and Z_2 are two stochastic processes and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i(Z_1, Z_2, t)dt + \sigma_i(Z_1, Z_2, t)dX_i \quad \text{with} \quad \sigma_i \geq 0, \quad i = 1, 2,$$

where $dX_i, i = 1, 2$, are the Wiener processes and $\mathrm{E}\left[dX_1 dX_2\right] = \rho_{12} dt$ with $\rho_{12} \in [-1, 1]$. Suppose that at $t = 0$, $(Z_1, Z_2) \in \Omega$. **<u>Show</u>** that in order to guarantee $(Z_1, Z_2) \in \Omega$ for any time $t \in [0, T]$, we need to require, for any $t \in [0, T]$ and for any point on Γ, the following condition to be held:

i. if $n_1 \neq 0$ and $n_2 = 0$, then

$$\begin{cases} n_1 \mu_1 \leq 0, \\ \sigma_1 = 0; \end{cases}$$

ii. if $n_1 = 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_2 \mu_2 \leq 0, \\ \sigma_2 = 0; \end{cases}$$

iii. if $n_1 \neq 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_1 \mu_1 + n_2 \mu_2 \leq 0, \\ n_1 \sigma_1 - \mathrm{sign}(n_1 n_2) n_2 \sigma_2 = 0, \quad \text{and} \quad \rho_{12} = -\mathrm{sign}(n_1 n_2), \end{cases}$$

where

$$\mathrm{sign}(n_1 n_2) = \begin{cases} 1, & \text{if } n_1 n_2 > 0, \\ -1, & \text{if } n_1 n_2 < 0. \end{cases}$$

If a point is a corner point, then there are two normals and we need to require this condition to be held for the two outer normal vectors.

(b) (2 points) Suppose that the domain Ω is $Z_{1l} \leq Z_1 \leq 1$ and $Z_{2l} \leq Z_2 \leq Z_1$, where Z_{1l} and Z_{2l} are constants, and $Z_{1l} \geq Z_{2l}$. **<u>Find</u>** the concrete condition **<u>according to the conditions given in (a)</u>** for each segment of the boundary.