**MATH 6202** **Test II (Part I)** **Spring 2009**

Name :______________________________

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## Show the details of your work !!

1. Consider the following problem:

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0) S \dfrac{\partial \overline{V}}{\partial S} - r\overline{V} = 0, \quad 0 \le S, \quad t \le T, \\ \overline{V}(S,T) = \begin{cases} \varphi_1(S), & 0 \le S \le B, \\ \varphi_2(S), & B < S, \end{cases} \end{cases}$$

where $\varphi_1(S)$ and $\varphi_2(S)$ are continuous functions and $\varphi_1(B) = \varphi_2(B)$ may not hold.

(a) Show that if between $\varphi_1(S)$ and $\varphi_2(S)$, the relation

$$\varphi_1(S) = -\left(\frac{B}{S}\right)^{2\left(r - D_0 - \sigma^2/2\right)/\sigma^2} \varphi_2\left(\frac{B^2}{S}\right),$$

or

$$\varphi_2(S) = -\left(\frac{B}{S}\right)^{2\left(r - D_0 - \sigma^2/2\right)/\sigma^2} \varphi_1\left(\frac{B^2}{S}\right)$$

hold, then $\overline{V}(B,t) = 0$.

(b) Based on the result above, show that for the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S \le B_u, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S \le B_u, \\ V(B_u,t) = 0, & t \le T, \end{cases}$$

the solution is

$$V(S,t) = e^{-r(T-t)} \int_0^{B_u} V_T(S') G_1(S',T;S,t,B_u)\, dS',$$

where

$$\begin{aligned} G_1(S',T;S,t,B_u) &= G(S',T;S,t) \\ &\quad - (B_u/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} G\left(S',T;B_u^2/S,t\right). \end{aligned}$$

Here

$$\begin{aligned} & G(S',T;S,t) \\ = \; & \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} e^{-\left[\ln(S'/S) - \left(r - D_0 - \sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}. \end{aligned}$$

2. Define

$$\mathbf{L}_{SAt} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} + \frac{S-A}{t}\frac{\partial}{\partial A} - r.$$

(a) Find the function of location of the free boundary at $t = T$, $S = S_f(A, T)$, for the LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_{SAt}V, V - \max(\alpha S - A, 0)\right) = 0, & 0 \le S,\ 0 \le A, \\ & t \le T, \\ V(S, A, T) = \max(\alpha S - A, 0), & 0 \le S,\ 0 \le A. \end{cases}$$

(b) Find the function of location of the free boundary at $t = T$, $A = A_f(S, T)$, for the LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_{SAt}V, V - \max(A - E, 0)\right) = 0, & 0 \le S,\ 0 \le A, \\ & t \le T, \\ V(S, A, T) = \max(A - E, 0), & 0 \le S,\ 0 \le A. \end{cases}$$

3. Suppose that $V(S_1, S_2, t)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_1^2 S_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \dfrac{1}{2}\sigma_2^2 S_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ +(r - D_{01})S_1\dfrac{\partial V}{\partial S_1} + (r - D_{02})S_2\dfrac{\partial V}{\partial S_2} - rV = 0, \\ \qquad\qquad\qquad 0 \le S_1,\ 0 \le S_2,\ 0 \le t \le T, \\ V(S_1, S_2, T) = \min(S_0, S_1, S_2), \qquad 0 \le S_1,\ 0 \le S_2. \end{cases}$$

Let $\xi_{02} = \dfrac{S_0 e^{-r(T-t)}}{S_2 e^{-D_{02}(T-t)}} = \dfrac{S_0 e^{-(r-D_{02})(T-t)}}{S_2}$, $\xi_{12} = \dfrac{S_1 e^{-D_{01}(T-t)}}{S_2 e^{-D_{02}(T-t)}} = \dfrac{S_1 e^{-(D_{01}-D_{02})(T-t)}}{S_2}$, and $V_2(\xi_{02}, \xi_{12}, t) = \dfrac{V(S_1, S_2, t)}{S_2 e^{-D_{02}(T-t)}}$. Derive the final-value problem for $V_2(\xi_{02}, \xi_{12}, t)$.

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## Show the details of your work !!

4. Suppose that $a(r,t) = a_0(t) + a_1(t)r$ and $b(r,t) = b_0(t) + b_1(t)r$. Show that the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + a(r,t)\dfrac{\partial^2 V}{\partial r^2} + b(r,t)\dfrac{\partial V}{\partial r} - rV = 0, \qquad 0 \le t \le T, \\ V(r,T) = 1 \end{cases}$$

has a solution in the form

$$V(r,t) = e^{A(t) - rB(t)}$$

with $A(T) = B(T) = 0$ and determine the system of ordinary differential equations the functions $A(t)$ and $B(t)$ should satisfy.

5. (a) $\mathbf{S}$ is a random vector and its covariance matrix is $\mathbf{B}$. Let $\bar{\mathbf{S}} = \mathbf{AS}$, $\mathbf{A}$ being a constant matrix, and its covariance matrix be $\mathbf{C}$. Find the relation among $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

   (b) How do we choose $\mathbf{A}$ so that $\mathbf{C}$ will be a diagonal matrix?

   (c) Suppose that $\bar{S}_1, \bar{S}_2, \cdots, \bar{S}_K$ are variables and $\bar{S}_{K+1}, \bar{S}_{K+2}, \cdots, \bar{S}_N$ are fixed numbers. Find the dependence of $S_{K+1}, S_{K+2}, \cdots, S_N$ on $S_1$, $S_2$, $\cdots$, $S_K$.

6. (a) Suppose that there is a domain $\Omega$ on the $(Z_1, Z_2)$-plane, the boundary of $\Omega$ is $\Gamma$, and $(n_1, n_2)^T$ is the outer normal vector of the boundary $\Gamma$. Assume that $Z_1$ and $Z_2$ are two stochastic processes and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i(Z_1, Z_2, t)dt + \sigma_i(Z_1, Z_2, t)dX_i \quad \text{with} \quad \sigma_i \ge 0, \quad i = 1, 2,$$

   where $dX_i, i = 1, 2,$ are the Wiener processes and $\mathrm{E}\left[dX_1 dX_2\right] = \rho_{12}dt$ with $\rho_{12} \in [-1, 1]$. Suppose that at $t = 0$, $(Z_1, Z_2) \in \Omega$. Show that in order to guarantee $(Z_1, Z_2) \in \Omega$ for any time $t \in [0, T]$, we need to require, for any $t \in [0, T]$ and for any point on $\Gamma$, the following condition to be held:

   i. if $n_1 \ne 0$ and $n_2 = 0$, then
$$\begin{cases} n_1\mu_1 \le 0, \\ \sigma_1 = 0; \end{cases}$$

   ii. if $n_1 = 0$ and $n_2 \ne 0$, then
$$\begin{cases} n_2\mu_2 \le 0, \\ \sigma_2 = 0; \end{cases}$$

iii. if $n_1 \neq 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_1\mu_1 + n_2\mu_2 \leq 0, \\ n_1\sigma_1 - \text{sign}(n_1 n_2) n_2 \sigma_2 = 0, \quad \text{and} \quad \rho_{12} = -\text{sign}(n_1 n_2), \end{cases}$$

where

$$\text{sign}(n_1 n_2) = \begin{cases} 1, & \text{if } n_1 n_2 > 0, \\ -1, & \text{if } n_1 n_2 < 0. \end{cases}$$

If a point is a corner point, then there are two normals and we need to require this condition to be held for the two outer normal vectors.

(b) Suppose that the domain $\Omega$ is $Z_{1l} \leq Z_1 \leq 1$ and $Z_{2l} \leq Z_2 \leq Z_1$, where $Z_{1l}$ and $Z_{2l}$ are constants, and $Z_{1l} \geq Z_{2l}$. Find the concrete condition corresponding to the condition given in a) on each segment of the boundary.