MATH 6202/8202 **Test II (Part A)** **Spring 2010**

Name :___________________________________

ID :___________________________________

Show the details of your work !!

1. (6 points) Consider the following problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)S\dfrac{\partial V}{\partial S}+\dfrac{S}{K}\sum_{i=1}^{K}\delta(t-t_i)\dfrac{\partial V}{\partial I}-rV=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0\le S,\quad 0\le I,\quad t\le T, \\ V(S,I,T)=\max(\pm(\alpha S-I),0),\qquad 0\le S,\quad 0\le I. \end{cases}$$

Let $\eta=\dfrac{I}{S}$ and $W=\dfrac{V}{S}$. In this case $W=W(\eta,t)$. **<u>Derive</u>** the PDE and the final condition for $W(\eta,t)$.

2. (7 points) Consider the following problem:

$$\begin{cases} \dfrac{\partial \overline{W}}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 \overline{W}}{\partial \eta^2}+(D_0-r)\,\eta\dfrac{\partial \overline{W}}{\partial \eta}-D_0\overline{W}=0,\quad 0\le\eta,\quad t\le T, \\ \overline{W}(\eta,T)=\begin{cases}\varphi_1(\eta), & 0\le\eta\le 1,\\ \varphi_2(\eta), & 1<\eta,\end{cases} \end{cases}$$

where $\varphi_1(\eta)$ and $\varphi_2(\eta)$ are continuous functions, and $\varphi_1(1)=\varphi_2(1)$ may not hold. **<u>Show</u>** that if

$$\begin{cases} \varphi_1(1)=\varphi_2(1), \\ \dfrac{d\varphi_1(\eta)}{d\eta}=\eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2}\dfrac{d\varphi_2(1/\eta)}{d\eta}, \end{cases}$$

then $\dfrac{\partial\overline{W}(1,t)}{\partial\eta}=0$.

3. (7 points) Suppose that $V(S_1,S_2,t)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma_1^2S_1^2\dfrac{\partial^2 V}{\partial S_1^2}+\rho_{12}\sigma_1\sigma_2S_1S_2\dfrac{\partial^2 V}{\partial S_1\partial S_2}+\dfrac{1}{2}\sigma_2^2S_2^2\dfrac{\partial^2 V}{\partial S_2^2} \\ +(r-D_{01})S_1\dfrac{\partial V}{\partial S_1}+(r-D_{02})S_2\dfrac{\partial V}{\partial S_2}-rV=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0\le S_1,\ 0\le S_2,\ 0\le t\le T, \\ V(S_1,S_2,T)=\max(S_0,S_1,S_2),\qquad 0\le S_1,\ 0\le S_2. \end{cases}$$

Let $\xi_{21}=\dfrac{S_2e^{-D_{02}(T-t)}}{S_1e^{-D_{01}(T-t)}}=\dfrac{S_2e^{-(D_{02}-D_{01})(T-t)}}{S_1}$, $\xi_{01}=\dfrac{S_0e^{-r(T-t)}}{S_1e^{-D_{01}(T-t)}}=\dfrac{S_0e^{-(r-D_{01})(T-t)}}{S_1}$, and $V_1(\xi_{21},\xi_{01},t)=\dfrac{V(S_1,S_2,t)}{S_1e^{-D_{01}(T-t)}}$. **<u>Derive</u>** the final-value problem for $V_1(\xi_{21},\xi_{01},t)$ and write the problem in such a form that only the volatilities of ξ_{21}, ξ_{01} and the correlation coefficient between the Wiener processes associated with the lognormal variables ξ_{21} and ξ_{01} appear in the PDE.

MATH 6202/8202 **Test II (Part B)** **Spring 2010**

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Show the details of your work !!

4. (6 points) Describe a way to determine the market price of risk for the spot interest rate.

5. (6 points) Assume that Z_1, Z_2, Z_3 are random variables and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i \left(Z_1, Z_2, Z_3, t\right) dt + \sigma_i \left(Z_1, Z_2, Z_3, t\right) dX_i, \quad i = 1, 2, 3,$$

where dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{i,j} dt$ with $-1 \le \rho_{i,j} \le 1$. In order to guarantee that if a point is in a domain Ω at time t^*, then the point is still in the domain Ω at $t = t^* + dt$ for a positive dt, it is necessary to require that the condition

$$n_1 dZ_1 + n_2 dZ_2 + n_3 dZ_3 \le 0$$

holds at any point on the boundary of the domain Ω, where n_1, n_2, and n_3 are the three components of the outer normal vector of the boundary at the point. Suppose that the domain Ω is $\{Z_{1,l} \le Z_1 \le 1, \quad Z_{2,l} \le Z_2 \le Z_1, \quad Z_{3,l} \le Z_3 \le Z_2\}$. **Show** that on the surfaces $Z_1 = 1$, $Z_2 = Z_{2,l}$, and $Z_3 = Z_2$, the condition is equivalent to $\{\mu_1 \le 0, \sigma_1 = 0\}$, $\{\mu_2 \ge 0, \sigma_2 = 0\}$, and $\{-\mu_2 + \mu_3 \le 0, \sigma_2 = \sigma_3, \rho_{2,3} = 1\}$, respectively.

6. The price of a convertible bond is the solution of the following linear complementarity problem on the domain $[0, \infty) \times [r_l, r_u] \times [0, T]$ in the $\{S, r, t\}$-space:

$$\begin{cases} \min\left(-\dfrac{\partial B_c}{\partial t} - \mathbf{L_{s,r}} B_c - kZ, \quad B_c(S,r,t) - nS\right) = 0, \\ B_c(S,r,T) = \max(Z, nS) \ge nS, \end{cases}$$

where

$$\mathbf{L_{s,r}} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S w \frac{\partial^2}{\partial S \partial r} + \frac{1}{2} w^2 \frac{\partial^2}{\partial r^2} + (r - D_0) S \frac{\partial}{\partial S} + (u - \lambda w)\frac{\partial}{\partial r} - r.$$

   (a) (6 points) Assuming that $D_0 > 0$ and there is only one free boundary, **find** the formulation of the corresponding free-boundary problem.

   (b) (2 points) **Does** $\dfrac{\partial B_c(S,r,t)}{\partial r} = 0$ **hold** on the free boundary in the formulation you give? Why?