MATH 6202/8202 **Test II (Part A)** **Spring 2011**

Name :______________________________

ID :______________________________

Show the details of your work !!

1. (6 points) As we know, the price of a European lookback strike put option, $V(S,H,t)$ is determined by the problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)S\dfrac{\partial V}{\partial S}-rV=0, & 0\le S\le H,\quad t\le T,\\ V(S,H,T)=\max(H-\beta S,0), & 0\le S\le H,\\ \dfrac{\partial V(S,S,t)}{\partial H}=0, & 0\le S,\quad t\le T, \end{cases}$$

in which there are three independent variables: S,H,t. **Reduce** this problem to a problem in which only two independent variables are involved.

2. **Show** the following two results which are related to the down-and-out call options:

   (a) (3.5 points) If $S\ge B_l$ and $S'\ge B_l$, then

$$\begin{aligned} & G_1\left(S',T;S,t,B_l\right)\\ = \;& G\left(S',T;S,t\right)-\left(B_l/S\right)^{2(r-D_0-\sigma^2/2)/\sigma^2}G\left(S',T;B_l^2/S,t\right)\ge 0, \end{aligned}$$

   where

$$\begin{aligned} & G\left(S',T;S,t\right)\\ = \;& \frac{1}{S'\sigma\sqrt{2\pi\left(T-t\right)}}e^{-\left[\ln(S'/S)-\left(r-D_0-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}. \end{aligned}$$

   (Hint: First it should be shown that this inequality is equivalent to the following inequalities:

$$\ln G\left(S',T;S,t\right)\ge\ln\left[\left(B_l/S\right)^{2(r-D_0-\sigma^2/2)/\sigma^2}G\left(S',T;B_l^2/S,t\right)\right]$$

   and

$$\left(\ln\frac{S'}{B_l}+\ln\frac{S}{B_l}\right)^2\ge\left(\ln\frac{S'}{B_l}-\ln\frac{S}{B_l}\right)^2.$$

)

   (b) (3.5 points) Let $c_o(S,t)$ and $C_o(S,t)$ be the prices of the European and American down-and-out call options, respectively. Between them the following is true:

$$C_o(S,t)\ge c_o(S,t)\quad\text{ for any } t.$$

3. (7 points) Consider the following problem

$$\begin{cases} \dfrac{\partial \overline{V}_1(\mathbf{y},\tau)}{\partial \tau} = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \rho_{ij} \dfrac{\partial^2 \overline{V}_1(\mathbf{y},\tau)}{\partial y_i \partial y_j}, & -\infty < \mathbf{y} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}(\mathbf{y},0) = V_{1T}(\mathbf{y}), & -\infty < \mathbf{y} < \infty, \end{cases}$$

where

$$V_{1T}(\mathbf{y}) \equiv V_T\left(e^{\sigma_1 y_1/\sqrt{2}}, e^{\sigma_2 y_2/\sqrt{2}}, \cdots, e^{\sigma_n y_n/\sqrt{2}}\right).$$

(a) Let

$$\mathbf{x} = \mathbf{R}\mathbf{y}$$

and

$$\overline{V}_2(\mathbf{x},\tau) = \overline{V}_1(\mathbf{y},\tau),$$

where $\mathbf{R}$ is a contant matrix:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}.$$

<u>Find</u> the equation and initial condition for $\overline{V}_2(\mathbf{x},\tau)$.

(b) **<u>Find</u>** $\mathbf{R}$ such that $\overline{V}_2(\mathbf{x},\tau)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{V}_2(\mathbf{x},\tau)}{\partial \tau} = \sum\limits_{l=1}^{n} \dfrac{\partial^2 \overline{V}_2(\mathbf{x},\tau)}{\partial x_l^2}, & -\infty < \mathbf{x} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}_2(\mathbf{x},0) = V_{2T}(\mathbf{x}), & -\infty < \mathbf{x} < \infty, \end{cases}$$

where $V_{2T}(\mathbf{x}) \equiv V_{1T}(\mathbf{R}^{-1}\mathbf{x})$.

MATH 6202/8202 **Test II (Part B)** **Spring 2011**

Name :______________________________

ID :______________________________

Show the details of your work !!

4. (6 points) Suppose that $a(r,t) = a_0(t) + a_1(t)r$ and $b(r,t) = b_0(t) + b_1(t)r$. **Show** that the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + a(r,t)\dfrac{\partial^2 V}{\partial r^2} + b(r,t)\dfrac{\partial V}{\partial r} - rV = 0, \qquad 0 \le t \le T, \\ V(r,T) = 1 \end{cases}$$

has a solution in the form

$$V(r,t) = e^{A(t) - rB(t)}$$

with $A(T) = B(T) = 0$ and **determine** the system of ordinary differential equations the functions $A(t)$ and $B(t)$ should satisfy.

5. (a) (3.5 points) **Show** that the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w)\dfrac{\partial V}{\partial r} - rV = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad t \le T, \\ V(r,T;T) = 1, \qquad\qquad r_l \le r \le r_u \end{cases}$$

is the same as that of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w)\dfrac{\partial V}{\partial r} - rV + \delta(t-T) = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad t \le T, \\ V(r,T^+;T) = 0, \qquad\qquad r_l \le r \le r_u \end{cases}$$

for any $t < T$.

(b) (3.5 points) Consider the following two procedures. The first one is to solve the problem:

$$\begin{cases} \dfrac{\partial V_{s1}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{s1}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{s1}}{\partial r} - rV_{s1} \\ \qquad + \displaystyle\sum_{k=1}^{2N} \delta(t - T - k/2) = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad T \le t \le T+N, \\ V_{s1}(r, T+N) = 0, \qquad\qquad r_l \le r \le r_u. \end{cases}$$

The second is to solve the following problems

$$\begin{cases} \dfrac{\partial V_{s1k}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{s1k}}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_{s1k}}{\partial r} - rV_{s1k} = 0, & r_l \le r \le r_u, \\ & T \le t \le T+k/2, \\ V_{s1k}(r, T+k/2) = 1, & r_l \le r \le r_u, \end{cases}$$

$k = 1, 2, \cdots, 2N$, and then obtain $\sum\limits_{k=1}^{2N} V_{s1k}(r,T)$ by adding $V_{s1k}(r,T)$, $k = 1, 2, \cdots, 2N$, together. **Explain** (i) why $V_{s1}(r,T) = \sum\limits_{k=1}^{2N} V_{s1k}(r,T)$ holds and (ii) in order to obtain the value of $V_{s1}(r,T)$ which procedure is better and why?

6. (a) (4 points) **Show** that under the transformation

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \end{cases}$$

the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial Z_i \partial Z_j} + r\sum_{i=1}^{2} Z_i\frac{\partial V}{\partial Z_i} - rV = 0$$

becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{2} b_i\frac{\partial V}{\partial \xi_i} - rV = 0,$$

where

$$\begin{cases} b_1 = \dfrac{rZ_1}{1 - Z_{1,l}}, \\ b_2 = \dfrac{r\left(Z_2 - Z_1\xi_2\right)}{Z_1 - Z_{2,l}} + \dfrac{\sigma_1\left(\sigma_1\xi_2 - \sigma_2\rho_{1,2}\right)}{\left(Z_1 - Z_{2,l}\right)^2}, \end{cases}$$

and $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\rho}_{1,2}$, are determined by

$$\begin{cases} \tilde{\sigma}_1^2 = \dfrac{\sigma_1^2}{\left(1 - Z_{1,l}\right)^2}, \\ \tilde{\sigma}_2^2 = \dfrac{\sigma_1^2\xi_2^2 - 2\sigma_1\sigma_2\xi_2\rho_{1,2} + \sigma_2^2}{\left(Z_1 - Z_{2,l}\right)^2}, \\ \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2} = \dfrac{\sigma_1\left(\sigma_2\rho_{1,2} - \sigma_1\xi_2\right)}{\left(1 - Z_{1,l}\right)\left(Z_1 - Z_{2,l}\right)}. \end{cases}$$

(b) (1 point) **Show** that the expression of b_2 can be rewritten as

$$b_2 = \frac{r\left(Z_2 - Z_1\xi_2\right)}{Z_1 - Z_{2,l}} - \frac{\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}\left(1 - Z_{1,l}\right)}{Z_1 - Z_{2,l}}.$$

(c) (2 points) b_i given above is a function of ξ_1, ξ_2, t and let $b_i(\xi_1, \xi_2, t)$ denote this function, $i = 1$ and 2. **Show** that if

$$\begin{cases} \tilde{\sigma}_1(0, \xi_2, t) = \tilde{\sigma}_1(1, \xi_2, t) = 0, & 0 \leq \xi_2 \leq 1, \\ \tilde{\sigma}_2(\xi_1, 0, t) = \tilde{\sigma}_2(\xi_1, 1, t) = 0, & 0 \leq \xi_1 \leq 1, \end{cases}$$

then

$$\begin{cases} b_1(0, \xi_2, t) \geq 0, \quad b_1(1, \xi_2, t) = 0, & 0 \leq \xi_2 \leq 1, \\ b_2(\xi_1, 0, t) \geq 0, \quad b_2(\xi_1, 1, t) = 0, & 0 \leq \xi_1 \leq 1. \end{cases}$$

(Hint: $r(\xi_1, \xi_2, t)|_{\xi_1=1} = 0$. This can be explained as follows. $\xi_1 = 1$ means $Z_1 = 1$, thus the zero-coupon bond curve must be flat near $T = 0$ and its derivative with respect to T at $T = 0$, $r(\xi_1, \xi_2, t)|_{\xi_1=1}$ equals zero.)