MATH 6202/8202 **Test II (Part A)** **Spring 2012**

Name :___________________________________

ID :___________________________________

Show the details of your work !!

1. (a) (3.5 points) Let $p_{\max}(S_1, S_2, t)$, $c_{\max}(S_1, S_2, t)$, and $\bar{c}_{\max}(S_1, S_2, t)$ be the prices of the three European options with payoff functions

$$\max(E - \max(S_1, S_2), 0), \quad \max(\max(S_1, S_2) - E, 0) \quad \text{and} \quad \max(S_1, S_2),$$

respectively. **Show**

$$p_{\max}(S_1, S_2, t) = Ee^{-r(T-t)} - \bar{c}_{\max}(S_1, S_2, t) + c_{\max}(S_1, S_2, t).$$

(b) (3.5 points) Let $\mathbf{P}$ be a positive definite matrix. As we know, in this case there exist a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{P} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where all the components of $\mathbf{\Lambda}$ are positive and $\mathbf{Q}$ satisfies the conditions $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det\mathbf{Q} > 0$. Let $\mathbf{y}$ and $\mathbf{y}_0$ be two vectors and define $\mathbf{R} = \mathbf{\Lambda}^{-1/2}\mathbf{Q}^T$, $\mathbf{x} = \mathbf{R}\mathbf{y}$, $\mathbf{x}_0 = \mathbf{R}\mathbf{y}_0$, and $\eta = \dfrac{\mathbf{y}_0 - \mathbf{y}}{\sqrt{2\tau}}$. **Show**

$$\det \mathbf{R} = \frac{1}{\sqrt{\det \mathbf{P}}} \quad \text{and} \quad \frac{(\mathbf{x}_0 - \mathbf{x})^T (\mathbf{x}_0 - \mathbf{x})}{4\tau} = \frac{\eta^T \mathbf{P}^{-1} \eta}{2}.$$

2. Let $V(S, A, t)$ be the price of a European Asian option with continuous arithmetic averaging, where A is the average of the price during the time period $[0, t]$. As we know, the equation for European Asian option with continuous arithmetic averaging is

$$\frac{\partial W(\eta, t)}{\partial t} + \mathbf{L}_{a,t} W(\eta, t) = 0,$$

where $W = V(S, A, t)/S$, $\eta = A/S$ and $\mathbf{L}_{a,t}$ is the time-dependent operator related to Asian options and given by

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2}{\partial \eta^2} + \left[(D_0 - r)\eta + \frac{1 - \eta}{t}\right] \frac{\partial}{\partial \eta} - D_0.$$

(a) (2 points) **Write** down the LC problem for an American Asian put option with a continuous arithmetic average strike price.

(b) (2 points) **Determine** where the PDE can always be used and a free boundary cannot appear and where a free boundary may appear.

(c) (2 points) **Derive** the free-boundary problem for this case. (Assume that there exists at most one free boundary.)

3. (7 points) Suppose that $V(S_1, S_2, t)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_1^2 S_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \dfrac{1}{2}\sigma_2^2 S_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ +(r - D_{01}) S_1 \dfrac{\partial V}{\partial S_1} + (r - D_{02}) S_2 \dfrac{\partial V}{\partial S_2} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S_1,\ 0 \le S_2,\ 0 \le t \le T, \\ V(S_1, S_2, T) = \max(S_0, S_1, S_2), \qquad 0 \le S_1,\ 0 \le S_2. \end{cases}$$

Define $S_0^* = S_0 e^{-r(T-t)}$, $S_i^* = S_i e^{-D_{0i}(T-t)}$, $i = 1, 2$. Let

$$\begin{aligned} \xi_{02} &= S_0^*/S_2^* = S_0 e^{-(r-D_{02})(T-t)}/S_2, \\ \xi_{12} &= S_1^*/S_2^* = S_1 e^{-(D_{01}-D_{02})(T-t)}/S_2, \\ V_2(\xi_{02}, \xi_{12}, t) &= V(S_1, S_2, t)/S_2^* = V(S_1, S_2, t)/(S_2 e^{-D_{02}(T-t)}). \end{aligned}$$

Show that $V_2(\xi_{02}, \xi_{12}, t)$ is the solution of the following problem:

$$\begin{cases} \dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma_{02}^2 \xi_{02}^2 \dfrac{\partial^2 V_2}{\partial \xi_{02}^2} + \rho_{012}\sigma_{02}\sigma_{12}\xi_{02}\xi_{12} \dfrac{\partial^2 V_2}{\partial \xi_{02}\partial \xi_{12}} \\ \quad + \dfrac{1}{2}\sigma_{12}^2 \xi_{12}^2 \dfrac{\partial^2 V_2}{\partial \xi_{12}^2} = 0, \qquad 0 \le \xi_{02}, \quad 0 \le \xi_{12}, \quad 0 \le t \le T, \\ V_2(\xi_{02}, \xi_{12}, T) = \max(1, \xi_{02}, \xi_{12}), \qquad 0 \le \xi_{02}, \quad 0 \le \xi_{12}, \end{cases}$$

where

$$\begin{aligned} \sigma_{02} &= \sigma_2, \\ \sigma_{12} &= \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}, \end{aligned}$$

and

$$\rho_{012} = \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma_{12}}.$$

MATH 6202/8202 **Test II (Part B)** **Spring 2012**

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Show the details of your work !!

4. (6 points) Suppose that any European-style interest rate derivative satisfies the equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV+f(t)=0, \quad r_l \le r \le r_u,$$

where all the coefficients in the equation are known. The value of N-year swap at time T is given by

$$V_s(T;r_s)=Q\left[1-Z(T;T+N)-\frac{r_s}{2}\sum_{k=1}^{2N}Z(T;T+k/2)\right],$$

where Q is the notional principal, r_s is the N-year swap rate and $Z(T;T+k/2)$ is the value of zero-coupon bond with maturity $k/2$ at time T. **Describe** how to find the price of a swaption with exercise swap rate r_{se} and maturity T, including to find $Z(T;T+N)$ and $\sum_{k=1}^{2N} Z(T;T+k/2)$, by solving this equation from $T+N$ to T twice and from T to 0 once.

5. (7 points) Consider the problem of the system of ordinary differential equations

$$\begin{cases} \dfrac{dA}{dt}=\mu B, \\ \dfrac{dB}{dt}=\dfrac{1}{2}\alpha B^2+\gamma B-1 \end{cases}$$

with the conditions

$$A(T,T)=0$$

and

$$B(T,T)=0.$$

Find the solution of the above problem of ordinary differential equations by solving the two ODEs, and **show** that your expressions of A and B can be rewritten as

$$\begin{cases} A=\ln\left(\dfrac{2\psi e^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)e^{\psi(T-t)}-(\gamma-\psi)}\right)^{2\mu/\alpha}, \\ B=\dfrac{2(e^{\psi(T-t)}-1)}{(\gamma+\psi)e^{\psi(T-t)}-(\gamma-\psi)} \\ \quad \text{with } \psi=\sqrt{\gamma^2+2\alpha} \end{cases}$$

if your solution is not in this form. (This problem is related to the Cox–Ingersoll–Ross interest rate model.)

6. (a) (5.0 points) Suppose that there is a domain Ω on the (Z_1, Z_2)-plane, the boundary of Ω is Γ, and $(n_1, n_2)^T$ is the outer normal vector of the boundary Γ. Assume that Z_1 and Z_2 are two stochastic processes and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i(Z_1, Z_2, t)dt + \sigma_i(Z_1, Z_2, t)dX_i \quad \text{with} \quad \sigma_i \geq 0, \quad i = 1, 2,$$

where $dX_i, i = 1, 2$, are the Wiener processes and $\mathrm{E}\left[dX_1 dX_2\right] = \rho_{12}dt$ with $\rho_{12} \in [-1, 1]$. Suppose that at $t = 0$, $(Z_1, Z_2) \in \Omega$. **Show** that in order to guarantee $(Z_1, Z_2) \in \Omega$ for any time $t \in [0, T]$, we need to require, for any $t \in [0, T]$ and for any point on Γ, the following condition to be held:

i. if $n_1 \neq 0$ and $n_2 = 0$, then

$$\begin{cases} n_1\mu_1 \leq 0, \\ \sigma_1 = 0; \end{cases}$$

ii. if $n_1 = 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_2\mu_2 \leq 0, \\ \sigma_2 = 0; \end{cases}$$

iii. if $n_1 \neq 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_1\mu_1 + n_2\mu_2 \leq 0, \\ n_1\sigma_1 - \mathrm{sign}(n_1 n_2) n_2 \sigma_2 = 0, \quad \text{and} \quad \rho_{12} = -\mathrm{sign}(n_1 n_2), \end{cases}$$

where

$$\mathrm{sign}(n_1 n_2) = \begin{cases} 1, & \text{if } n_1 n_2 > 0, \\ -1, & \text{if } n_1 n_2 < 0. \end{cases}$$

If a point is a corner point, then there are two normals and we need to require this condition to be held for the two outer normal vectors.

(b) (2.0 points) Suppose that the domain Ω is $Z_{1l} \leq Z_1 \leq 1$ and $Z_{2l} \leq Z_2 \leq Z_1$, where Z_{1l} and Z_{2l} are constants, and $Z_{1l} \geq Z_{2l}$. **Find** the concrete condition for each segment of the boundary according to the condition given in a).