Homework Problems (Part II) and Projects for "Derivative Securities and Difference Methods"

5

Basic Numerical Methods

Problems

1. Suppose $x_m = m\Delta x$.
   a) Find the order of the error of the following approximate function

$$u(x) \approx \frac{x_{m+1} - x}{\Delta x} u(x_m) + \frac{x - x_m}{\Delta x} u(x_{m+1})$$

   by the Taylor expansion. Here $x \in [x_m, x_{m+1}]$.
   b) Find the order of the error of the following approximate function

$$\begin{aligned} u(x) \approx & \frac{(x - x_m)(x - x_{m+1})}{2\Delta x^2} u(x_{m-1}) \\ & - \frac{(x - x_{m-1})(x - x_{m+1})}{\Delta x^2} u(x_m) \\ & + \frac{(x - x_{m-1})(x - x_m)}{2\Delta x^2} u(x_{m+1}) \end{aligned}$$

   by the Taylor expansion. Here $x \in [x_{m-1}, x_{m+1}]$.
2. Show that from

$$\begin{cases} a_{m+1} = a_m + b_m h_m + c_m h_m^2 + d_m h_m^3, \\ b_{m+1} = b_m + 2c_m h_m + 3d_m h_m^2, \\ c_{m+1} = c_m + 3d_m h_m, \\ \qquad m = 0, 1, \cdots, M-2, \end{cases}$$

   and

$$\begin{cases} a_M = a_{M-1} + b_{M-1} h_{M-1} + c_{M-1} h_{M-1}^2 + d_{M-1} h_{M-1}^3, \\ c_M = c_{M-1} + 3d_{M-1} h_{M-1}, \end{cases}$$

   the following relation can be derived:

$$\begin{aligned}&h_{m-1}c_{m-1}+2(h_{m-1}+h_m)c_m+h_mc_{m+1}\\=&\frac{3(a_{m+1}-a_m)}{h_m}-\frac{3(a_m-a_{m-1})}{h_{m-1}},\\&\qquad m=1,2,\cdots,M-1.\end{aligned}$$

3. Consider the cubic spline problem. Suppose that the derivative is given at $x=x_{M}$, insteady of assuming $c_{M}=0$. Derive the equation which should replace the equation $c_{M}=0$ in the system for $c_0,c_1,\cdots,c_{M}$.
4. Suppose $x_m=m\Delta x$, $y_l=l\Delta y$, and $\tau^n=n\Delta\tau$. Find the expression of the error of each of the following approximations:
   a) $u(x_m,\tau^{n+1/2})\approx\dfrac{u(x_m,\tau^{n+1})+u(x_m,\tau^n)}{2}$;
   b) $\dfrac{\partial u}{\partial\tau}(x_m,\tau^n)\approx\dfrac{u(x_m,\tau^{n+1})-u(x_m,\tau^n)}{\Delta\tau}$;
   c) $\dfrac{\partial u}{\partial\tau}(x_m,\tau^{n+1/2})\approx\dfrac{u(x_m,\tau^{n+1})-u(x_m,\tau^n)}{\Delta\tau}$;
   d) $\dfrac{\partial u}{\partial x}(x_m,\tau^n)\approx\dfrac{u(x_{m+1},\tau^n)-u(x_m,\tau^n)}{\Delta x}$;
   e) $\dfrac{\partial u}{\partial x}(x_m,\tau^n)\approx\dfrac{u(x_{m+1},\tau^n)-u(x_{m-1},\tau^n)}{2\Delta x}$;
   f) $\dfrac{\partial u}{\partial x}(x_m,\tau^n)\approx\dfrac{3u(x_m,\tau^n)-4u(x_{m-1},\tau^n)+u(x_{m-2},\tau^n)}{2\Delta x}$;
   g) $\dfrac{\partial^2 u}{\partial x^2}(x_m,\tau^n)\approx\dfrac{u(x_{m+1},\tau^n)-2u(x_m,\tau^n)+u(x_{m-1},\tau^n)}{\Delta x^2}$;
   h)
$$\begin{aligned}\frac{\partial^2 u}{\partial x\partial y}(x_m,y_l,\tau^n)\approx\frac{1}{2\Delta x}\Bigg[&\frac{u(x_{m+1},y_{l+1},\tau^n)-u(x_{m+1},y_{l-1},\tau^n)}{2\Delta y}\\&-\frac{u(x_{m-1},y_{l+1},\tau^n)-u(x_{m-1},y_{l-1},\tau^n)}{2\Delta y}\Bigg].\end{aligned}$$

5. For $y\in[-1,1]$, we define
$$T_{N}(y)=\cos\left(N\cos^{-1}y\right),$$
where N is an integer. Let
$$y_j=\cos\frac{j\pi}{N},\quad j=0,1,\cdots,N.$$
Show
   a) $T_{k+1}(y)-2yT_k(y)+T_{k-1}(y)=0$, $k\geq1$.
   b) $T_{N}(y)$ is a polynomial of degree N for any nonnegative integer.
   c) $$\frac{dT_{N}(y_j)}{dy}=\begin{cases}N^2, & j=0,\\ 0, & j=1,2,\cdots,N-1,\\ (-1)^{N+1}N^2, & j=N;\end{cases}$$

d) $$\frac{d^2T_N(y_j)}{dy^2} = \begin{cases} \dfrac{N^2\left(N^2-1\right)}{3}, & j=0, \\ \dfrac{(-1)^{j+1}N^2}{\left(1-y_j^2\right)}, & j=1,2,\cdots,N-1, \\ \dfrac{(-1)^N N^2\left(N^2-1\right)}{3}, & j=N; \end{cases}$$

e) $$\frac{d^3T_N(y_j)}{dy^3} = \frac{(-1)^{j+1}3N^2y_j}{\left(1-y_j^2\right)^2},\ j=1,2,\cdots,N-1.$$

6. Let
$$h_j(y) = \frac{(-1)^{j+1}\left(1-y^2\right)T_N'(y)}{c_jN^2(y-y_j)},\quad j=0,1,\cdots,N,$$
where $T_N(y)$ is the Chebyshev polynomial of first kind with degree N, $y_j=\cos\frac{j\pi}{N}$, $j=0,1,\cdots,N$, , and
$$c_j = \begin{cases} 2,\ j=0, \\ 1,\ j=1,2,\cdots,N-1, \\ 2,\ j=N. \end{cases}$$

a) Show
$$h_j(y_i) = \frac{(-1)^{j+1}\left(1-y_i^2\right)T_N'(y_i)}{c_jN^2(y_i-y_j)} = \delta_{ij},\quad i,j=0,1,\cdots,N,$$
where δ_{ij} is the Kronecker delta.

b) Define
$$d_{ij} = \frac{dh_j(y_i)}{dy},\quad i,j=0,1,\cdots,N.$$
Show that
$$d_{ij} = \begin{cases} \dfrac{(-1)^{i+j}c_i}{c_j(y_i-y_j)}, & i\neq j, \\ \dfrac{2N^2+1}{6}, & i=j=0, \\ -\dfrac{y_j}{2\left(1-y_j^2\right)}, & i=j=1,2,\cdots,N-1, \\ -\dfrac{2N^2+1}{6}, & i=j=N. \end{cases}$$

c) Let $f_1(y_j)$ denote the values of the function $f_1(y)$ at $y=y_j$, $j=0,1,\cdots,N$. Show that

$$p_{N1}(y) = \sum_{j=0}^{N} h_j(y) f_1(y_j)$$

is an interpolation polynomail with degree N for $f_1(y)$ on $[-1,1]$ and

$$\frac{dp_{N1}(y_i)}{dy} = \sum_{j=0}^{N} d_{ij} f_1(y_j).$$

d) Define $x = (1-y)/2$ or $y = 1-2x$. Let $f(x_j)$ denote the values of the function $f(x)$ at $x = x_j$, $j = 0, 1, \cdots, N$. Show that

$$p_N(x) = \sum_{j=0}^{N} h_j(1-2x) f(x_j)$$

is an interpolation polynomail with degree N for $f(x)$ on $[0,1]$ and

$$\frac{dp_N(x_i)}{dx} = \sum_{j=0}^{N} D_{ij} f(x_j),$$

where

$$D_{ij} = \begin{cases} \dfrac{(-1)^{i+j} c_i}{c_j (x_i - x_j)}, & i \neq j, \\ -\dfrac{2N^2+1}{3}, & i = j = 0, \\ \dfrac{1-2x_j}{4x_j (1-x_j)}, & i = j = 1, 2, \cdots, N-1, \\ \dfrac{2N^2+1}{3}, & i = j = N. \end{cases}$$

7. Derive the formulae of the LU decomposition method for the following almost tridiagonal system

$$\mathbf{A}\mathbf{x} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & d_1 \\ a_2 & b_2 & c_2 & & 0 & d_2 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & a_{m-1} & b_{m-1} & d_{m-1} \\ & & & & a_m & d_m \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}.$$

8. Suppose that we already have a solver for solving tridiagonal system:
$$\mathbf{A}\mathbf{x} = \mathbf{q},$$
where
$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} \\ & & & a_m & b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{m-1} \\ q_m \end{bmatrix}.$$
In order to solve the following almost tridiagonal system
$$\mathbf{A}\mathbf{x} = \mathbf{q},$$
where
$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} \\ & & a_m & b_m & c_m \end{bmatrix} \quad \text{or} \quad \mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} \\ & & a_m & b_m & c_m \end{bmatrix},$$
we can convert it to a tridiagonal system and solve the new system by the existing solver. Design such a method.
9. Describe the Jacobi iteration, the Gauss–Seidel iteration, and the method of successive over relaxation for an $n \times n$ system of linear equations.
10. Suppose $f(x) = 0$ is a nonlinear equation. Derive the iteration formulae of Newton's method and the secant method for solving the nonlinear equation.
11. a) For each of the following methods, describe the details of the method and its advantage and disadvantage:
    i. The secant method;
    ii. The bisection method;
    iii. The modified secant method.

    b) Based on the methods in a), design an efficient and robust method of finding a root of the equation $f(x) = 0$.
12. Suppose
$$\mathbf{A}_1 = \mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)}) + \frac{\mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(0)}) - \mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)})(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})}{\left\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right\|_2^2}(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})^T.$$
Show that the following relations hold:
$$\mathbf{A}_1(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}) = \mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(0)})$$
and
$$\mathbf{A}_1\mathbf{z} = \mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)})\mathbf{z} \quad \text{whenever} \quad (\mathbf{x}^{(1)} - \mathbf{x}^{(0)})^T\mathbf{z} = 0.$$

13. Prove that if $\mathbf{A} \in \mathbb{R}^{n\times n}$ is nonsingular, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\mathbf{y}^T\mathbf{A}^{-1}\mathbf{x} \neq -1$, then $\mathbf{A} + \mathbf{x}\mathbf{y}^T$ is also nonsingular, moreover,
$$(\mathbf{A} + \mathbf{x}\mathbf{y}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{x}\mathbf{y}^T\mathbf{A}^{-1}}{1 + \mathbf{y}^T\mathbf{A}^{-1}\mathbf{x}}.$$
14. a) Show
$$\mathbf{H}_m\mathbf{x} = \beta\mathbf{e}_1,$$
where
$$\begin{aligned} \mathbf{x} &= [x_1, x_2, \cdots, x_m]^T, \\ \beta &= \sqrt{\mathbf{x}^T\mathbf{x}}, \\ \mathbf{H}_m &= \mathbf{I}_m - \frac{1}{\beta(\beta - x_1)}\mathbf{u}\mathbf{u}^T, \\ &\mathbf{u} \text{ being } \mathbf{x} - \beta\mathbf{e}_1. \end{aligned}$$
b) Using the result in a), design a method to obtain an orthogonal matrix $\mathbf{Q}$ from $\mathbf{A}$ such that $\mathbf{A} = \mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix with nonnegative diagonals.
15. Let f_m^n denote $f(m\Delta x, n\Delta\tau)$. Find the truncation error of the explicit difference scheme
$$\begin{aligned} \frac{u_m^{n+1} - u_m^n}{\Delta\tau} = a_m^n & \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} \\ & + b_m^n \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} + c_m^n u_m^n \end{aligned}$$
to the parabolic partial differential equation
$$\frac{\partial u}{\partial \tau} = a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial u}{\partial x} + c(x,\tau)u.$$
16. Show that the truncation error of the Crank–Nicloson scheme for the heat equation at the point $(x_m, \tau^{n+1/2})$ is in the following form:
$$\Delta\tau^2 \left[\frac{1}{24}\frac{\partial^3 u}{\partial\tau^3}(x_m, \eta^{(1)}) - \frac{a}{8}\frac{\partial^4 u}{\partial x^2\partial\tau^2}(x_m, \eta^{(2)})\right] - \frac{\Delta x^2 a}{12}\frac{\partial^4 u}{\partial x^4}(\xi, \eta^{(3)}),$$
where $\xi \in (x_{m-1}, x_{m+1})$, $\eta^{(k)} \in (\tau^n, \tau^{n+1})$, $k = 1, 2, 3$, and a is the conductivity coefficient in the heat equation.
17. Let f_m^n denote $f(m\Delta x, n\Delta\tau)$. Find the truncation error of the implicit difference scheme
$$\begin{aligned} \frac{u_m^{n+1} - u_m^n}{\Delta\tau} = & \frac{a_m^{n+1/2}}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right) \\ & + \frac{b_m^{n+1/2}}{2}\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right) \\ & + \frac{c_m^{n+1/2}}{2}(u_m^{n+1} + u_m^n) \end{aligned}$$

to the parabolic partial differential equation

$$\frac{\partial u}{\partial \tau} = a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial u}{\partial x} + c(x,\tau)u.$$

18. The heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

can also be discretized by

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \theta\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right)$$

or

$$u_m^{n+1} - \theta\alpha(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) = u_m^n + (1-\theta)\alpha(u_{m+1}^n - 2u_m^n + u_{m-1}^n),$$

where $0 \le \theta \le 1$ and $\alpha = \Delta\tau/\Delta x^2$. This scheme is called the θ–scheme. It is clear that when $\theta = 0$, the scheme reduces to the explicit scheme and when $\theta = 1/2$, the scheme becomes the Crank–Nicolson scheme. Show that the order of truncation error of the θ–scheme is

$$O\left((1-2\theta)\,\Delta\tau + \Delta\tau^2 + \Delta x^2\right).$$

(Hint: Discretize the partial differential equation at $x = x_m$ and $\tau = \tau^{n+\theta}$.)

19. Consider the three-point explicit finite-difference scheme:

$$u_m^{n+1} = a_m u_{m-1}^n + b_m u_m^n + c_m u_{m+1}^n, \quad m = 0, 1, \cdots, M,$$

where $a_m \ge 0$, $b_m = 1 - a_m - c_m \ge 0$, $c_m \ge 0$ and $a_0 = c_M = 0$. Show

$$\max_{0\le m\le M} |u_m^{n+1}| \le \max_{0\le m\le M} |u_m^n|.$$

This means that the numerical procedure is stable under the maximum norm.

20. a) Consider an $M \times M$ matrix

$$\mathbf{A} = \begin{pmatrix} a & b & 0 & \cdots\cdots & 0 & b \\ b & a & b & 0 & \cdots\cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots\ \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots\ 0 \\ 0 & \cdots\cdots & 0 & b & a & b \\ b & 0 & \cdots\cdots & 0 & b & a \end{pmatrix}.$$

Suppose $a = q + 2/h^2$ and $b = -1/h^2$. Show that its eigenvalues are $\lambda_j = q + \dfrac{4}{h^2}\sin^2\dfrac{\theta_j}{2}$, $j = 0, 1, \cdots, M-1$, where $\theta_j = j\frac{2\pi}{M}$, and the corresponding eigenvectors are

$$\mathbf{v}_j = \begin{pmatrix} 1 \\ \cos\theta_j \\ \cos 2\theta_j \\ \vdots \\ \cos(M-1)\,\theta_j \end{pmatrix}, \quad j = 0, 1, \cdots, \mathrm{int}\left(\frac{M}{2}\right),$$

and

$$\mathbf{v}_j = \begin{pmatrix} 0 \\ \sin\theta_j \\ \sin 2\theta_j \\ \vdots \\ \sin(M-1)\,\theta_j \end{pmatrix}, \quad j = \mathrm{int}\left(\frac{M}{2}\right) + 1, \cdots, M-1,$$

respectively, where $\mathrm{int}\left(\dfrac{M}{2}\right)$ is the integer part of $\dfrac{M}{2}$.

b) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

c) Suppose $a = \dfrac{q}{2} + \dfrac{2}{h^2}$ and $b = \dfrac{q}{4} - \dfrac{1}{h^2}$, find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{A}^{-1}$.

21. Consider the explicit scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}, \quad m = 1, 2, \cdots, M-1$$

with $u_0^{n+1} = f_l(\tau^{n+1})$ and $u_M^{n+1} = f_u(\tau^{n+1})$. Determine when it is stable with respect to initial values in L_2 norm and when it is unstable. (Suppose $a > 0$.)

22. Consider the implicit scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right),$$
$$m = 1, 2, \cdots, M-1$$

with $u_0^{n+1} = f_l(\tau^{n+1})$ and $u_M^{n+1} = f_u(\tau^{n+1})$. Show that it is always stable with respect to initial values in L_2 norm. (Suppose $a > 0$.)

23. By using the von Neumann method, show that for periodic problems, the θ–scheme (see Problem 18) is stable for all $\alpha > 0$ if $\dfrac{1}{2} \le \theta \le 1$ and that it is stable for $0 < \alpha \le \dfrac{1}{2(1-2\theta)}$ if $0 < \theta < \dfrac{1}{2}$.

24. Consider the following parabolic partial differential equation:

$$\frac{\partial u}{\partial \tau} = a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y},$$

where $a_{11}(x, y, \tau) \geq 0, a_{22}(x, y, \tau) \geq 0, a_{12}(x, y, \tau) = \rho_{12}(x, y, \tau)\sqrt{a_{11}a_{12}}$ with $\rho_{12} \in [-1, 1]$, and b_1, b_2 are any functions of x, y, τ. This equation can be approximated by

i)

$$\begin{aligned}
&\frac{u_{m,n}^{k+1} - u_{m,n}^{k}}{\Delta\tau} \\
&= \frac{a_{11,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1} - 2u_{m,n}^{k+1} + u_{m-1,n}^{k+1}}{\Delta x^2} + \frac{u_{m+1,n}^{k} - 2u_{m,n}^{k} + u_{m-1,n}^{k}}{\Delta x^2}\right) \\
&\quad + a_{12,m,n}^{k+\frac{1}{2}}\left(\frac{u_{m+1,n+1}^{k+1} - u_{m+1,n-1}^{k+1} - u_{m-1,n+1}^{k+1} + u_{m-1,n-1}^{k+1}}{4\Delta x\Delta y}\right. \\
&\qquad\qquad \left. + \frac{u_{m+1,n+1}^{k} - u_{m+1,n-1}^{k} - u_{m-1,n+1}^{k} + u_{m-1,n-1}^{k}}{4\Delta x\Delta y}\right) \\
&\quad + \frac{a_{22,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1} - 2u_{m,n}^{k+1} + u_{m,n-1}^{k+1}}{\Delta y^2}\right. \\
&\qquad\qquad \left. + \frac{u_{m,n+1}^{k} - 2u_{m,n}^{k} + u_{m,n-1}^{k}}{\Delta y^2}\right) \\
&\quad + \frac{b_{1,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1} - u_{m-1,n}^{k+1}}{2\Delta x} + \frac{u_{m+1,n}^{k} - u_{m-1,n}^{k}}{2\Delta x}\right) \\
&\quad + \frac{b_{2,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1} - u_{m,n-1}^{k+1}}{2\Delta y} + \frac{u_{m,n+1}^{k} - u_{m,n-1}^{k}}{2\Delta y}\right)
\end{aligned}$$

or

ii)

$$\begin{aligned}
&\frac{u_{m,n}^{k+1} - u_{m,n}^{k}}{\Delta\tau} \\
&= \frac{a_{11,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1} - 2u_{m,n}^{k+1} + u_{m-1,n}^{k+1}}{\Delta x^2} + \frac{u_{m+1,n}^{k} - 2u_{m,n}^{k} + u_{m-1,n}^{k}}{\Delta x^2}\right) \\
&\quad + a_{12,m,n}^{k+\frac{1}{2}}\left(\frac{u_{m+1,n+1}^{k+1} - u_{m+1,n-1}^{k+1} - u_{m-1,n+1}^{k+1} + u_{m-1,n-1}^{k+1}}{4\Delta x\Delta y}\right. \\
&\qquad\qquad \left. + \frac{u_{m+1,n+1}^{k} - u_{m+1,n-1}^{k} - u_{m-1,n+1}^{k} + u_{m-1,n-1}^{k}}{4\Delta x\Delta y}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{a_{22,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1}-2u_{m,n}^{k+1}+u_{m,n-1}^{k+1}}{\Delta y^2}\right.\\
&\qquad\left.+\frac{u_{m,n+1}^{k}-2u_{m,n}^{k}+u_{m,n-1}^{k}}{\Delta y^2}\right)\\
&+\frac{b_{1,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{-u_{m+2,n}^{k+1}+4u_{m+1,n}^{k+1}-3u_{m,n}^{k+1}}{2\Delta x}\right.\\
&\qquad\left.+\frac{-u_{m+2,n}^{k}+4u_{m+1,n}^{k}-3u_{m,n}^{k}}{2\Delta x}\right)\\
&+\frac{b_{2,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{3u_{m,n}^{k+1}-4u_{m,n-1}^{k+1}+u_{m,n-2}^{k+1}}{2\Delta y}\right.\\
&\qquad\left.+\frac{3u_{m,n}^{k}-4u_{m,n-1}^{k}+u_{m,n-2}^{k}}{2\Delta y}\right)
\end{aligned}
$$

if $b_1(x,y,\tau)\geq 0$ and $b_2(x,y,\tau)\leq 0$. By the von Neumann method, show that they are stable.

(Hint:

a) First show that the amplification factor λ can be written as $\lambda = \dfrac{1+a+ib}{1-a-ib}$.

b) Then show that $|\lambda|^2\leq 1$ is equivalent to $|1-a-ib|^2-|1+a+ib|^2 = -4a\geq 0$.

c) Finally show $-4a\geq 0$ by using the following inequalities: i). $A^2+B^2+2\rho AB=(A+\rho B)^2+B^2\left(1-\rho^2\right)\geq 0$ if $|\rho|\leq 1$; ii). $\cos 2\theta-4\cos\theta+3=2(\cos\theta-1)^2\geq 0$.

)

25. Show that if

$$\max_{0\leq m\leq M}\frac{x_m^2(1-x_m)^2\bar{\sigma}_m^2}{2}\frac{\Delta\tau}{\Delta x^2}\leq\frac{1}{2},$$

then for the scheme with variable coefficients

$$
\begin{aligned}
\frac{u_m^{n+1}-u_m^n}{\Delta\tau} &= \frac{1}{2}[x_m(1-x_m)\bar{\sigma}_m]^2\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{\Delta x^2}\\
&\quad+(r-D_0)x_m(1-x_m)\frac{u_{m+1}^n-u_{m-1}^n}{2\Delta x}\\
&\quad-\left[r(1-x_m)+D_0x_m\right]u_m^n,
\end{aligned}
$$

the condition $|\lambda_\theta(x_m,\tau^n)|\leq 1+O(\Delta\tau)$ is satisfied for any $x_m=m/M\in[0,1]$. (When you prove this result, you should derive the stability condition for explicit schemes by yourself.)

26. For the scheme with variable coefficients

$$\begin{aligned}
&\frac{u_m^{n+1}-u_m^n}{\Delta\tau}\\
=&\frac{1}{4}[x_m(1-x_m)\bar{\sigma}_m]^2\left(\frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{\Delta x^2}+\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{\Delta x^2}\right)\\
&+\frac{1}{2}(r-D_0)x_m(1-x_m)\left(\frac{u_{m+1}^{n+1}-u_{m-1}^{n+1}}{2\Delta x}+\frac{u_{m+1}^n-u_{m-1}^n}{2\Delta x}\right)\\
&-\frac{1}{2}\left[r\left(1-x_m\right)+D_0x_m\right]\left(u_m^{n+1}+u_m^n\right),
\end{aligned}$$

show that the condition $|\lambda_\theta(x_m,\tau^n)| \le 1+O(\Delta\tau)$ is satisfied for any $x\in[0,1]$.

27. a) Consider the explicit scheme given in Problem 15. Assume that its stability with respect to initial value and nonhomogeneous term is proved under certain conditions. Show that for its solution, under these conditions there is the following relation: $u\left(x,\tau;\Delta x,\Delta\tau\right) = u(x,\tau)+a\left(x,\tau;\dfrac{\Delta x^2}{\Delta\tau}\right)\Delta\tau+O(\Delta\tau^2)$, where $\left|O(\Delta\tau^2)\right| \le c\Delta\tau^2$, c being bounded as $\Delta\tau\to 0$ with $\dfrac{\Delta x^2}{\Delta\tau} = constant$.

b) Suppose we have two such approximate solutions $u\left(x,\tau;\Delta x,\Delta\tau\right)$ and $u\left(x,\tau;\Delta x/2,\Delta\tau/4\right)$. Find a linear combination

$$(1-d)\times u\left(x,\tau;\Delta x,\Delta\tau\right)+d\times u\left(x,\tau;\Delta x/2,\Delta\tau/4\right)$$

such that it is an approximate solution with an error of $O(\Delta\tau^2)$.

28. a) Suppose we have two approximate solutions with errors of $O(\Delta\tau^2)$: $u\left(x,\tau;\dfrac{1}{12},\dfrac{T}{16}\right)$ and $u\left(x,\tau;\dfrac{1}{9},\dfrac{T}{12}\right)$. Find a linear combination

$$(1-d)\times u\left(x,\tau;\frac{1}{12},\frac{T}{16}\right)+d\times u\left(x,\tau;\frac{1}{9},\frac{T}{12}\right)$$

such that it is an approximate solution with an error of $O(\Delta\tau^3)$.

b) Suppose we further have another approximate solution with errors of $O(\Delta\tau^2)$: $u\left(x,\tau;\dfrac{1}{15},\dfrac{T}{20}\right)$. Find a linear combination

$$d_0\times u\left(x,\tau;\frac{1}{15},\frac{T}{20}\right)+d_1\times u\left(x,\tau;\frac{1}{12},\frac{T}{16}\right)+d_2\times u\left(x,\tau;\frac{1}{9},\frac{T}{12}\right)$$

such that it is an approximate solution with an error of $O(\Delta\tau^4)$, where $d_0=1-d_1-d_2$.

29. Assume that the volatility of a stock is a function of the stock price. Describe a method determining the function from the market data.

Projects

General Requirements

A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

B) Each code should use an input file to specify all the problem parameters and the computational parameters and an output file to store all the results. In an output file, the name of the problem, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

C) Submit results in form of tables. When a result is given, always provide the problem parameters and the computational parameters.

1. **Cumulative distribution function and Black–Scholes formulae**. Write three functions:

   a) **double** N(double z)

   for computing approximate values of the cumulative distribution function for the standardized normal variable, where z is the independent variable (see the footnote given in Subsection 2.4.2).
   - Give the values of $N(z)$ for $z = -2, -1, 0, 1, 2$.

   b) **double** BS(double S, double E, double tau, double r, double D0, double sigma, char option),

   which gives prices of the European options by using Black–Scholes formulae (see Subsection 2.4.4). When the value of the character 'option' is equal to 'c' or 'C', the value of the European call needs to be evaluated. Otherwise, the value of the European put needs to be evaluated.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$.

   c) **double** BS_bar(double xi, double E, double tau, double r, double D0, double sigma, char option)

This function gives the value of $\bar{c}(\xi,\tau) = c(S,t)/(S+E)$ or $\bar{p}(\xi,\tau) = p(S,t)/(S+E)$.

- For $\xi = 0.5128, 0.5000, 0.4878$, $E = 95, 100, 105$, $\tau = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, calculate the results of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ by this function.

2. **Quadratic interpolation and LU decomposition of a tridiagonal system**.
For the the quadratic interpolation method (see Subsection 5.1.1), write a function

a) **double** Interpolation(double x, int M, double *y)

Suppose that x, M, and $y_m = y(x_m), m = 0, 1, ..., M$, are given, where $x_m = m/M$. This function gives an approximate value of y(x) by quadratic interpolation. The concrete method is as follows. If $x < 1/2M$, then interpolate or extrapolate $y(x)$ by (x_0, y_0), (x_1, y_1), (x_2, y_2), if $x_m - 1/2M \le x < x_m + 1/2M$, $m = 1, 2, \cdots, M-1$, then interpolate $y(x)$ by (x_{m-1}, y_{m-1}), (x_m, y_m), (x_{m+1}, y_{m+1}), and if $x_M - 1/2M \le x$, then interpolate or extrapolate $y(x)$ by (x_{M-2}, y_{M-2}), (x_{M-1}, y_{M-1}), (x_M, y_M).

- Let $M = 5$ and the six components from y_0 to y_5 are 0.0, 0.008, 0.064, 0.216, 0.512, 1.0. Calculate the values of $y(x)$ for $x = -0.1, 0.45, 1.01$ by this function.

For LU decomposition, write two functions:

b) **int** LUT(int m, double *a, double *b, double *c, double *q, double *x).

Suppose that we have a tridiagonal system (5.10). The number of unknowns is given in the integer 'm.' The nonhomogeneous term q_i is given in q[i-1] (the i-th component of the array 'q'). The coefficients a_i, b_i, and c_i are given in the i-th component of the arrays 'a,' 'b,' and 'c,' respectively. Write a function to solve the system by using the method described in Subsection 5.2.1. If all the u_i are not equal to zero, then the code should return an integer number 0 and gives the value of the i-th unknown in the i-th component of the array x. If one of u_i is equal to zero, then the solution(s) of the system cannot be found by the method (or the system has no solution), and the code should return an integer number 1. The values in the arrays 'a', 'b', 'c', and 'q' are required unchanged.

- Let $m = 4$, $a_2 = a_3 = a_4 = -0.48$, $b_1 = b_2 = b_3 = b_4 = 1$, $c_1 = c_2 = c_3 = -0.49$, $q_1 = 0.02$, $q_2 = 0.05$, $q_3 = 0.08$, and $q_4 = 2.56$. Find the solution of the system (5.10).

c) **int** LUAT(int m, double *a, double *b, double *c, double *q, double *x)

This is a solver for an almost tridiagonal system by LU decomposition. The almost tridiagonal system is in the following form:

$$\mathbf{Ax} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & 0 & \\ & \ddots & \ddots & & \ddots & \\ & 0 & a_{m-1} & & b_{m-1} & c_{m-1} \\ & & a_m & & b_m & c_m \end{bmatrix}.$$

This function calculates $\mathbf{x}$ if m, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{q}$ are given. Require m, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{q}$ unchanged.

- Let $m = 5$, $\mathbf{a} = \{1.75, -0.48, -0.48, -0.48, 0.25\}$, $\mathbf{b} = \{-0.5, 1, 1, 1, -0.5\}$, $\mathbf{c} = \{0.25, -0.49, -0.49, -0.49, 1.75\}$, $\mathbf{q} = \{1.5, 0.05, 0.08, 0.11, 7.5\}$, calculate the result of $\mathbf{x}$ by this function.

6

Initial-Boundary Value and LC Problems

Problems

1. Suppose that we determine the price of an American vanilla call/put option through solving the following problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ \bar{\tau} \geq 0, \\ u(x,0) = g(x,0), & -\infty < x < \infty, \end{cases}$$

where
$$g(x,\bar{\tau}) = \max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}), 0\right).$$

Describe a numerical method for solving this problem by using an explicit scheme.

2. As we know, an American lookback strike put option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\left(\dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W\right),\ W - \max(\eta - \beta, 0)\right) = 0, & 1 \leq \eta, \quad t \leq T, \\ W(\eta, T) = \max(\eta - \beta, 0), & 1 \leq \eta, \\ \dfrac{\partial W}{\partial \eta}(1,t) = 0, & t \leq T, \end{cases}$$

where we assume that $\beta \geq 1$ and the operator $\mathbf{L}_\eta$ is defined by

$$\mathbf{L}_\eta \equiv \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + (D_0 - r)\,\eta\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem on $[0,1]$ and with an initial condition, and design an explicit method for solving the new problem.

3. Suppose that ψ is a binomial random variable and its two values are ψ_0 and ψ_1. Show the following:
   a) If $\mathrm{E}[\psi] = 0$ and $\mathrm{E}[\psi^2] = 1$, then $\psi_0\psi_1 = -1$.
   b) If $\mathrm{E}[\psi] = 0$ and $\psi_0\psi_1 = -1$, then $\mathrm{E}[\psi^2] = 1$.
   c) If $\mathrm{E}[\psi] = 0$ and $\psi_0\psi_1 = -1 + O(\Delta t)$, then $\mathrm{E}[\psi^2] = 1 + O(\Delta t)$.
4. a) Derive the binomial methods proposed by Cox, Ross, and Rubinstein and by McDonald.
   b) Can the parameter p in the Cox-Ross-Rubinstein binomial method always represent a probability? Find out when it can and when it cannot. Can the parameter p given in the book by McDonald always represent a probability? Find out when it can and when it cannot.
5. From the Black–Scholes equation, we know that when a derivative is priced, the value of the stock price at time t^n, S_n, and the value at time t^{n+1}, S_{n+1}, have the following relations:

$$\mathrm{E}_D\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$$

and

$$\mathrm{E}_D\left[S_{n+1}^2\right] = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2,$$

where $\Delta t = t^{n+1} - t^n$ (see Problem 22 of Chapter 2). Thus if the possible values of S_{n+1} are $S_{n+1,0} = S_n/u$ and $S_{n+1,1} = uS_n$, and the probabilities of S_{n+1} being $S_{n+1,0}$ and the probabilities of S_{n+1} being $S_{n+1,1}$ are $1-p$ and p, respectively. then in a binomial method, u and p should be determined by

$$\begin{cases} pu + (1-p)\, u^{-1} = \mathrm{e}^{(r-D_0)\Delta t}, \\ pu^2 + (1-p)\, u^{-2} = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t}. \end{cases}$$

If Δt is very small, this problem can be approximated by

$$\begin{cases} pu + (1-p)\, u^{-1} = 1 + (r-D_0)\Delta t, \\ pu^2 + (1-p)\, u^{-2} = 1 + [2(r-D_0)+\sigma^2]\Delta t. \end{cases}$$

   a) Find u and p for both cases (suppose $u > 1$).
   b) Consider a more general system

$$\begin{cases} pu + (1-p)\, u^{-1} = 1 + (r-D_0)\Delta t + O(\Delta t^2), \\ pu^2 + (1-p)\, u^{-2} = 1 + [2(r-D_0)+\sigma^2]\Delta t + O(\Delta t^2). \end{cases}$$

   Show that if u and p are determined by such a system, then we always have

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}} + O\left(\Delta t^{3/2}\right)$$

   and

$$p = \frac{1}{2}\left[1 + \frac{\sqrt{\Delta t}}{\sigma}\left(r - D_0 - \frac{1}{2}\sigma^2\right)\right] + O\left(\Delta t^{3/2}\right).$$

(Hint: When you derive the expression for p, write u as $1 + \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2 + c\Delta t^{3/2} + O(\Delta t^2)$, c being a constant.)

6. Describe the binomial methods for solving American vanilla call/put options
7. Show that the Cox-Ross-Rubinstein binomial method for European options almost is an explicit difference scheme for the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V}{\partial y^2} + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial V}{\partial y} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < y < \infty, \quad t \le T, \\ V(y,T) = \max(\pm(\mathrm{e}^y - 1), 0), \qquad -\infty < y < \infty, \end{cases}$$

where $y = \ln S$, S being the price of the stock and assume $E = 1$.

8. Show that the relation

$$V(S, t_i^-) = V(S - D_i(S), t_i^+)$$

becomes

$$\begin{aligned} &\overline{V}(\xi,\ \tau_i^+) \\ &= \left[1 - D_i\left(\frac{\xi P_m}{1-\xi}\right)\frac{1-\xi}{P_m}\right]\overline{V}\left(\frac{P_m\xi - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)},\ \tau_i^-\right) \end{aligned}$$

under the transformation

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ \tau = T - t, \\ \overline{V}(\xi,\tau) = \dfrac{V(S,t)}{S + P_m}. \end{cases}$$

9. Suppose that we determine the price of an American vanilla call/put option through solving the following problem:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(\pm(2\xi - 1), 0)\right) = 0, \quad 0 \le \xi \le 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tau \ge 0, \\ \overline{V}(\xi, 0) = \max(\pm(2\xi - 1), 0), \qquad\qquad 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

Describe a numerical method for solving this problem by using a second order implicit scheme. (Discuss the discretization of the problem only.)

10. As we know, an American average strike call option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{a,t}W,\ W(\eta,t) - \max(\alpha-\eta,0)\right) = 0, & 0 \le \eta,\ t \le T, \\ W(\eta,T) = \max(\alpha-\eta,0), & 0 \le \eta, \end{cases}$$

where $\alpha \approx 1$ and

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + \left[(D_0 - r)\eta + \frac{1-\eta}{t}\right]\frac{\partial}{\partial\eta} - D_0.$$

Convert this problem into a problem on a finite domain and with an initial condition, and design an implicit second-order method for solving this new problem. (Discuss the discretization of the problem only.)

11. Based on the partial differential equation

$$\frac{\partial\overline{V}}{\partial\tau} = \frac{1}{2}\bar\sigma^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2\overline{V}}{\partial\xi^2} + r\xi(1-\xi)\frac{\partial\overline{V}}{\partial\xi} - r(1-\xi)\overline{V},$$

design an implicit method for the LC problem of American options with discrete dividends.

12. Suppose that the scheme

$$\begin{aligned} &\frac{v_m^{n+1} - v_m^n}{\Delta\tau} \\ &= \frac{1}{4}\bar\sigma_m^2\xi_m^2(1-\xi_m)^2\left(\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{\Delta\xi^2} + \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\Delta\xi^2}\right) \\ &\quad + \frac{1}{2}(r - D_0)\xi_m(1-\xi_m)\left(\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2\Delta\xi} + \frac{v_{m+1}^n - v_{m-1}^n}{2\Delta\xi}\right) \\ &\quad - \frac{1}{2}[r(1-\xi_m) + D_0\xi_m](v_m^{n+1} + v_m^n) \end{aligned}$$

is used for solving an American option problem. Design a projeted direct method, which you think is most accurate, to find the solution at each time step.

13. Consider the following LC problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial\bar\tau} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar\tau) - g(x,\bar\tau)\right) = 0, & -\infty < x < \infty,\ \bar\tau \ge 0, \\ u(x,0) = g(x,0), & -\infty < x < \infty, \end{cases}$$

where

$$g(x,\bar{\tau}) = \max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}), 0\right).$$

Suppose an implicit finite-difference method based on such a formulation is used for solving an American option problem. Design an iteration method similar to the SOR method for a linear system to find the solution of the problem at each time step.

14. The heat equation

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial x^2}$$

can be approximated by the explicit first-order scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}$$

or the implicit second-order scheme (the Crank–Nicolson scheme)

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right).$$

When do we choose the explicit first-order scheme and when do we use the implicit second-order scheme? Why should we choose the implicit second-order scheme if we need highly accurate results?

15. a) Find a closed-form solution of the problem:

$$\begin{cases} \dfrac{\partial c_u}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_u}{\partial S^2} + (r-D_0)S\dfrac{\partial c_u}{\partial S} - rc_u = 0, \quad 0 \le S, \quad 0 \le t \le T, \\ c_u(S,T) = \begin{cases} \max(S-E,0), & \text{if} \quad 0 \le S < g(T), \\ 0, & \text{if} \quad g(T) \le S. \end{cases} \end{cases}$$

Here we assume $g(T) > E$.

b) Consider the following European barrier option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & \\ & f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S-E,0), & f(T) \le S \le g(T), \\ V(f(t),t) = 0, & 0 \le t \le T, \\ V(g(t),t) = 0, & 0 \le t \le T, \end{cases}$$

where $S = f(t)$ and $S = g(t)$ are the locations of the lower and upper barriers with $f(t) < E$ and $g(t) > E$. Assume that we need to find the solution by numerical methods. Design a SSM for this problem based on the result given in a) . (Here the problem can be defined on a non-rectangular domain.)

16. Suppose that η_1, η_2, and p are given, where $0 < \eta_1 < \eta_2$, $2\eta_2 - \eta_1 < 1$, and $p > 1$. Set $\eta_3 = 2\eta_2 - \eta_1$ and let $f(\eta)$ be a function on $[0, 1]$ satisfying the condition $f(0) = 0$ and its derivative be equal to

$$f_\eta(\eta) = \begin{cases} d, & 0 \le \eta < \eta_1, \\ a(\eta - \eta_2)^4 + b(\eta - \eta_2)^2 + c, & \eta_1 \le \eta < \eta_3, \\ d, & \eta_3 \le \eta \le 1. \end{cases}$$

Here $d = a(\eta_2 - \eta_1)^4 + b(\eta_2 - \eta_1)^2 + c$, which guarantees $f_\eta(\eta)$ is continuous at $\eta = \eta_1$. From the definition of η_3, we know $\eta_2 - \eta_1 = \eta_3 - \eta_2$, so $f_\eta(\eta)$ is also continuous at $\eta = \eta_3$.

a) Assume that the following three conditions hold:
   (i) $\dfrac{f_\eta(\eta_2)}{f_\eta(\eta_1)} = \dfrac{c}{a(\eta_2 - \eta_1)^4 + b(\eta_2 - \eta_1)^2 + c} = p$,
   (ii) $f_{\eta\eta}(\eta_1) = 4a(\eta_1 - \eta_2)^3 + 2b(\eta_1 - \eta_2) = 0$,
   (iii) $f(1) = 1$.
   Find the expressions of a, b, and c as functions of η_1, η_2, η_3, and p and show that $f(\eta)$ is an increasing function on $[0, 1]$ in this case.
b) Find the expression of $f(\eta)$.
c) When solving a PDE/OPE problem, a variable mesh can be realized by using transformation. Suppose that the independent variable in a PDE/ODE problem is η and a new variable is introduced by setting $\xi = f(\eta)$. How should we choose the parameters in the function $f(\eta)$ if we want to let the mesh size in the region near the point $\eta = 0.4$ is about $1/10$ of the mesh size in the regions $[0, 0.2]$ and $[0.6, 1]$?

17. Let $\bar{c}(\xi, \tau) = c(S, t)/(S + P_m)$ and $\bar{p}(\xi, \tau) = p(S, t)/(S + P_m)$, where $\xi = S/(S + P_m)$ and $\tau = T - t$. Derive the expressions of $\bar{c}(\xi, \tau)$ and $\bar{p}(\xi, \tau)$ and find the limits of $\bar{c}(\xi, \tau)$ and $\bar{p}(\xi, \tau)$ as ξ tends to 0 and 1. Also write down the formulae for the case $P_m = E$.

18. Suppose that $V(S, t)$ satisfies the following jump condition at $t = t_i$:

$$V(S, t_i^-) = V(S - D_i(S), t_i^+)$$

and that $V_0(S, t)$ is continuous at $t = t_i$. Define

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ \tau = T - t, \\ u(\xi, \tau) = \dfrac{V(S, t) - V_0(S, t)}{S + P_m}, \\ u_0(\xi, \tau) = \dfrac{V_0(S, t)}{S + P_m}, \end{cases}$$

where P_m is a positive number. Show that the following jump condition for $u(\xi,\tau)$ holds:

$$\begin{aligned} & u\left(\xi,\tau_i^+\right) \\ = & \left[1-\frac{1-\xi}{P_m}D_i\left(\frac{\xi P_m}{1-\xi}\right)\right]\left[u\left(\frac{P_m\xi - D_i\left(\frac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\frac{\xi P_m}{1-\xi}\right)(1-\xi)},\tau_i^-\right)\right. \\ & \left. +u_0\left(\frac{P_m\xi - D_i\left(\frac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\frac{\xi P_m}{1-\xi}\right)(1-\xi)},\tau_i\right)\right] - u_0\left(\xi,\tau_i\right). \end{aligned}$$

19. Design a SSM for European vanilla options with discrete dividends and a constant volatility, and formulate the problem as a problem defined on a finite domain and with an initial condition.
20. Design a SSM for Bermudan options with variable volatilities and formulate the problem as a problem defined on a finite domain and with an initial condition.
21. Suppose r and D_0 are constant and $\sigma = \sigma(S)$. Derive the symmetry relations for Bermudan options.
22. Find a transformation to convert an average price call option problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-D_0)S\dfrac{\partial V}{\partial S} + \dfrac{S}{T}\dfrac{\partial V}{\partial I} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le I, \quad t \le T, \\ V(S,I,T) = \max(I-E,0), \qquad 0 \le S, \quad 0 \le I, \end{cases}$$

where

$$I = \frac{1}{T}\int_0^t S(\tau)d\tau,$$

into the problem

$$\begin{cases} \dfrac{\partial f}{\partial \tau} - \dfrac{1}{2}\sigma^2\left[\xi - \dfrac{1}{(r-D_0)T}(1-\mathrm{e}^{-(r-D_0)\tau})\right]^2 \dfrac{\partial^2 f}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f(\xi,0) = \max(\xi,0), \qquad -\infty < \xi < \infty. \end{cases}$$

23. Find a closed-form solution of the problem:

$$\begin{cases} \dfrac{\partial \tilde f_0}{\partial \tau} - \dfrac{\sigma^2}{2(r-D_0)^2T^2}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)^2 \dfrac{\partial^2 \tilde f_0}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ \tilde f_0(\xi,0) = \max(\xi,0), \qquad -\infty < \xi < \infty. \end{cases}$$

24. Convert the problem

$$\begin{cases} \dfrac{\partial f_1}{\partial \tau} - \dfrac{1}{2}\sigma^2 \left[\xi - \dfrac{1}{(r-D_0)T}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)\right]^2 \dfrac{\partial^2 f_1}{\partial \xi^2} = \dfrac{\sigma^2 \xi\, \mathrm{e}^{-\xi^2/4\tau_1}}{4\sqrt{\pi\tau_1}} \\ \quad \times \left[\xi - \dfrac{2}{(r-D_0)T}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)\right], \qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f_1(\xi, 0) = 0, \qquad -\infty < \xi < \infty. \end{cases}$$

into a problem defined on $[0,1]$ and with an initial condition, and design an implicit second-order scheme for the new problem.

25. By using the transformation

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ r = r, \\ \tau = T - t, \\ u(\xi, r, \tau) = \dfrac{B_c(S,r,t)}{n\,(S+P_m)}, \end{cases}$$

the two-factor convertible bond problem for non-dividend stocks

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + \rho\sigma S w \dfrac{\partial^2 B_c}{\partial S \partial r} + \dfrac{1}{2} w^2 \dfrac{\partial^2 B_c}{\partial r^2} + rS\dfrac{\partial B_c}{\partial S} \\ \qquad + (u - \lambda w)\dfrac{\partial B_c}{\partial r} - rB_c + kZ = 0, \\ \qquad\qquad 0 \le S, \quad r_l \le r \le r_u, \quad 0 \le t \le T, \\ B_c(S, r, T) = \max(Z, nS), \qquad 0 \le S, \quad r_l \le r \le r_u \end{cases}$$

can be converted into a problem on a finite domain with a bounded final condition. The one-factor convertible zero-coupon bond problem for non-dividend stocks

$$\begin{cases} \dfrac{\partial b_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 b_c}{\partial S^2} + rS\dfrac{\partial b_c}{\partial S} - rb_c = 0, \quad 0 \le S, \quad 0 \le t \le T, \\ b_c(S, r, T) = \max(Z, nS), \qquad 0 \le S \end{cases}$$

has the following solution:

$$nc(S,t;Z/n) + \mathrm{e}^{-r(T-t)}Z,$$

where $c(S,t;Z/n)$ is the price of a call option with an exercise price Z/n. Find the partial differential equation and the final condition the difference between the two bonds should satisfy. Convert the derived problem into a problem on a finite domain and with an initial condition by using the transformation above, and briefly describe a second-order implicit scheme for the new problem.

26. Suppose that $c(S,\sigma,t)$ and $p(S,\sigma,t)$ are solutions of the following problems

$$\begin{cases} \dfrac{\partial c}{\partial t} + \mathbf{L}_{S,\sigma} c = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ c(S,\sigma,T) = \max(S-E,0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u \end{cases}$$

and

$$\begin{cases} \dfrac{\partial p}{\partial t} + \mathbf{L}_{S,\sigma} p = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ p(S,\sigma,T) = \max(E-S,0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where $\mathbf{L}_{S,\sigma}$ is an operator defined by

$$\mathbf{L}_{S,\sigma} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} + (r-D_0)S\frac{\partial}{\partial S} + (p-\lambda q)\frac{\partial}{\partial \sigma} - r.$$

Show that the following put–call parity relation

$$c(S,\sigma,t) - p(S,\sigma,t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$$

holds by the superposition principle. (Hint: Let u denote $c(S,\sigma,t) - p(S,\sigma,t)$. Show that u is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathbf{L}_{S,\sigma} u = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ u(S,\sigma,T) = S - E, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u \end{cases}$$

and that $S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$ is also the solution of this problem.)

27. Convert the following double moving barrier call option problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & \\ & f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S-E,0), & f(T) \le S \le g(T), \\ V(f(t),t) = 0, & 0 \le t \le T, \\ V(g(t),t) = g(t) - E, & 0 \le t \le T \end{cases}$$

into a problem that has a smooth solution and an initial condition, and design an implicit pseudo-spectral method for the new problem.

28. For the new problem obtained in Problem 25, design an implicit pseudo-spectral method.

Projects

General Requirements

A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the problem, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

C) If not specified, for each case two results are required. For the first result, a 20×12 mesh should be used.(In this case, the error of the solution might be quite large.) For the second result, the accuracy required is 0.01, and the mesh used should be as coarse as possible.

D) Submit results in form of tables or figures. When a result is given, always provide the problem parameters and the computational parameters.

1. **Explicit method (6.3)** Suppose that σ, r are constants and the dividends are given discretely or continuously. Write a code for European, Bermudan, and American calls and puts.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$ (see Subsection 6.3.3).
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $\sigma = 0.2$, and two dividend payments of \$1.25 paid at $t = 2$ months and $t = 8$ months. $D(S)$ is defined by
   $$D(S) = \begin{cases} S & \text{if } S \le d, \\ d & \text{if } S > d, \end{cases}$$
   where d is the dividend payment.
   - Taking the European call option with $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ as an example, show that the explicit method (6.3) is unstable if $\varDelta\tau$ is too large. For this problem, only one example is required. Plot the S-c curve with $t = 0$.
2. **Binomial methods: (6.28) with (6.25)–(6.27) and (6.28) with (6.18) and (6.23)** Suppose that σ, r, D_0 are constants. Write a code

for European, Bermudan, and American calls and puts. For this problem, instead of the result on a 20×12 mesh, a result with $\Delta t = T/12$ is required.

- For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.025$, and $\sigma = 0.2$.
- For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$.
- For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.

3. **Implicit method (6.45) (solving the corresponding system by direct methods)**. Suppose that σ, r, and D_0 are constants. Write a code for European, Bermudan, and American calls and puts.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.025$, and $\sigma = 0.2$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
4. **Singularity-separating implicit method with scheme (6.45)** Suppose that σ, r are constants and the dividends are given discretely or continuously. Write a code for Bermudan calls and puts with continuous dividends and a code for European vanilla calls and puts with discrete dividends. Calculate the difference between the value of the option and the closed-form solution of a corresponding European vanilla option numerically. In order to calculate the price of a Bermudan put, Compute a corresponding call first and then obtain the value of the Bermudan put by using the symmetry relation.
   - For Bermudan call options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$. For Bermudan put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $K = 12$.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $\sigma = 0.2$, and two dividend payments of \$1.25 paid at $t = 2$ months and $t = 8$ months. $D(S)$ is defined by
   $$D(S) = \begin{cases} S & \text{if } S \le d, \\ d & \text{if } S > d, \end{cases}$$
   where d is the dividend payment.

7

Free-Boundary Problems

Problems

1. Consider the following free-boundary problem that is related to American lookback strike put options:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left(D_0 - r\right)\eta\dfrac{\partial W}{\partial \eta} - D_0 W = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le \eta \le \eta_f\left(t\right), \quad 0 \le t \le T, & \\ W\left(\eta, T\right) = \max\left(\eta - \beta, 0\right), & 1 \le \eta \le \eta_f\left(T\right), \\ \dfrac{\partial W}{\partial \eta}\left(1, t\right) = 0, & 0 \le t \le T, \\ W\left(\eta_f, t\right) = \eta_f - \beta, & 0 \le t \le T, \\ \dfrac{\partial W}{\partial \eta}\left(\eta_f, t\right) = 1, & 0 \le t \le T, \\ \eta_f\left(T\right) = \beta\max\left(1, D_0/r\right). & \end{cases}$$

By using the solution of the problem

$$\begin{cases} \dfrac{\partial W_1}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W_1}{\partial \eta^2} + (D_0 - r)\eta\dfrac{\partial W_1}{\partial \eta} - D_0 W_1 = 0, & \eta \ge 1, \quad 0 \le t \le T, \\ W_1(\eta, T) = \max(\eta - \beta, 0), & \eta \ge 1, \\ \dfrac{\partial W_1}{\partial \eta}(1, t) = 0, & 0 \le t \le T, \end{cases}$$

convert this problem into a problem whose solution has a continuous derivative everywhere. Here we also require that the problem is defined on a rectangular domain: $[0, 1] \times [0, T]$ and with an initial condition. (Assume $1 < \beta$).

2. Consider the following free-boundary problem that is related to American average strike call options:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1-\eta}{t}\right]\dfrac{\partial W}{\partial \eta} - D_0 W = 0, & \\ & \eta_f(t) \le \eta, \quad t \le T, \\ W(\eta, T) = \max(1-\eta, 0), & \eta_f(T) \le \eta, \\ W(\eta_f(t), t) = 1 - \eta_f(t), & t \le T, \\ \dfrac{\partial W}{\partial \eta}(\eta_f(t), t) = -1, & t \le T, \\ \eta_f(T) = \min\left(1, \ \dfrac{1 + D_0 T}{1 + rT}\right). & \end{cases}$$

Convert this problem into a problem with a singularity weaker than the singularity here for $t > 0$. Also require that the new problem is defined on a rectangular domain, has an initial condition and the right boundary corresponds to the free-boundary boundary.

3. Let $C(S, \sigma, t;\ a, b, c, d)$ and $P(S, \sigma, t;\ a, b, c, d)$ denote the prices of American two-factor call and put options and $S_{cf}(\sigma, t;\ a, b, c, d)$ and $S_{pf}(\sigma, t; a, b, c, d)$ be their optimal exercise prices. Here, a, b, c, and d are parameters (or parameter functions) for the risk-free interest rate r, dividend yield rate D_0, correlation coefficient ρ, and market price of volatility risk λ, respectively. Show that between American two-factor put and call options there is the following put–call symmetry relation:

$$\begin{cases} P(S, \sigma, t; a, b, c, d) = \dfrac{S}{E} C\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right), \\ C(S, \sigma, t; a, b, c, d) = \dfrac{S}{E} P\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right), \\ S_{pf}(\sigma, t; a, b, c, d) \times S_{cf}(\sigma, t; b, a, -c, d - c\sigma) = E^2. \end{cases}$$

4. Consider the following free-boundary problem that is related to American call options:

$$
\begin{cases}
\dfrac{\partial V}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^2(1-\xi)^2\dfrac{\partial^2 V}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\dfrac{\partial V}{\partial \xi} & \\
\qquad -[r(1-\xi)+D_0\xi]V, & 0 \le \xi < \xi_f(\tau), \quad 0 \le \tau, \\
V(\xi,0) = \max(2\xi-1,0), & 0 \le \xi < \xi_f(0), \\
V\left(\xi_f(\tau),\tau\right) = 2\xi_f(\tau)-1, & 0 \le \tau, \\
\dfrac{\partial V}{\partial \xi}\left(\xi_f(\tau),\tau\right) = 2, & 0 \le \tau, \\
\xi_f(0) = \max\left(\dfrac{1}{2}, \dfrac{r}{r+D_0}\right). &
\end{cases}
$$

a) Convert this problem into a problem whose solution has a continuous derivative everywhere. Here we also require that the problem is defined on a rectangular domain and with an initial condition.
b) Design a second-order implicit method to solve the new problem.

5. Consider the following the free-boundary problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial \tau} = \dfrac{1}{2}\sigma^2\left[\xi + \dfrac{1}{\bar{\eta}_f - 1}\right]^2 \dfrac{\partial^2 u}{\partial \xi^2} & \\
\qquad + \left[(D_0 - r)\left(\xi + \dfrac{1}{\bar{\eta}_f - 1}\right) + \dfrac{\xi}{\bar{\eta}_f - 1}\dfrac{d\bar{\eta}_f}{d\tau}\right]\dfrac{\partial u}{\partial \xi} - D_0 u, & 0 \le \xi \le 1, \\
& 0 \le \tau \le T, \\
u(\xi,0) = 0, & 0 \le \xi \le 1, \\
\dfrac{\partial u}{\partial \xi}(0,\tau) = 0, & 0 \le \tau \le T, \\
u(1,\tau) = \bar{\eta}_f(\tau) - \beta - W_1(\bar{\eta}_f(\tau), T-\tau), & 0 \le \tau \le T, \\
\dfrac{\partial u}{\partial \xi}(1,\tau) = (\bar{\eta}_f(\tau) - 1)\left[1 - \dfrac{\partial W_1(\bar{\eta}_f(\tau), T-\tau)}{\partial \eta}\right], & 0 \le \tau \le T, \\
\bar{\eta}_f(T) = \beta \max\left(1, D_0/r\right), &
\end{cases}
$$

where $W_1(\eta, T-\tau)$ is a given function. Design a second-order implicit method to solve this problem which is the new problem obtained in Problem 1.

6. Consider the nonlinear system consisting of the following equations

$$
\begin{aligned}
&\frac{u_m^{n+1} - u_m^n}{\Delta\tau} \\
&= \frac{1}{2}\left[k_2 m^2\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) + \frac{k_1 m}{2}\left(u_{m+1}^{n+1} - u_{m-1}^{n+1}\right) - k_0 u_m^{n+1}\right]
\end{aligned}
$$

$$+\frac{1}{2}\left[k_2 m^2\left(u_{m+1}^n-2u_m^n+u_{m-1}^n\right)+\frac{k_1 m}{2}\left(u_{m+1}^n-u_{m-1}^n\right)-k_0 u_m^n\right]$$
$$+\frac{s_f^{n+1}-s_f^n}{\left(s_f^{n+1}+s_f^n\right)\Delta\tau}\left[\frac{m}{2}\left(u_{m+1}^{n+1}-u_{m-1}^{n+1}\right)+\frac{m}{2}\left(u_{m+1}^n-u_{m-1}^n\right)\right],$$
$$m=0,1,2,\cdots,M-1,$$

and

$$u_{M}^{n+1}=g(s_f^{n+1},\tau^{n+1}),$$
$$\frac{3u_{M}^{n+1}-4u_{M-1}^{n+1}+u_{M-2}^{n+1}}{2\Delta\xi}=h\left(s_f^{n+1},\tau^{n+1}\right),$$

where u_m^n are known, τ^{n+1} is given, k_0, k_1, and k_2 are constants, and $g(s,\tau)$ and $h(s,\tau)$ are given functions. Discuss how to solve this system, provide at least two methods that you think are simple and effective, and give the details for one of the methods.

7. Is the extrapolation technique always helpful and why?
8. Why the extrapolation technique can still be used when a non-uniform mesh in τ: $\tau^n=n^2T/N^2, n=0,1,\cdots,N$, is used?
9. Design an exponential scheme to approximate

$$a(\xi)\frac{d^2U}{d\xi^2}+b(\xi)\frac{dU}{d\xi}+c(\xi)U,$$

where $a(\xi)\ge 0$ and $c(\xi)\le 0$.

10. Assume σ to be a random variable satisfying

$$d\sigma=p(\sigma,t)dt+q(\sigma,t)dX,$$

where dX is a Wiener process. In this case, evaluating American call options can be reduced to solving the following free-boundary problem:

$$\begin{cases}\dfrac{\partial C}{\partial t}+\mathbf{L}_{\mathbf{s},\sigma}C=0, & 0\le S\le S_f(\sigma,t),\\ & \sigma_l\le\sigma\le\sigma_u,\quad 0\le t\le T,\\ C(S,\sigma,T)=\max(S-E,0), & 0\le S\le S_f(\sigma,T),\\ & \sigma_l\le\sigma\le\sigma_u,\\ C\left(S_f(\sigma,t),\sigma,t\right)=S_f(\sigma,t)-E, & \sigma_l\le\sigma\le\sigma_u,\quad 0\le t\le T,\\ \dfrac{\partial C\left(S_f(\sigma,t),\sigma,t\right)}{\partial S}=1, & \sigma_l\le\sigma\le\sigma_u,\quad 0\le t\le T,\\ S_f(\sigma,T)=\max(E,rE/D_0), & \sigma_l\le\sigma\le\sigma_u,\end{cases}$$

where

$$\mathbf{L}_{\mathbf{S},\sigma} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2}{\partial \sigma^2}$$
$$+(r - D_0)S\frac{\partial}{\partial S} + (p - \lambda q)\frac{\partial}{\partial \sigma} - r.$$

a) Convert this problem into a problem defined on a rectangular domain and whose solution has a singularity weaker than the singularity here.
b) Design a second-order implicit method to solve the new problem. (Here and also for c), do not require to discuss the solution of the nonlinear system.)
c) Design a pseudo-spectral method to solve the new problem.

11. Consider the following free-boundary problem related to one-factor convertible bonds:

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0)S\dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, \\ \qquad\qquad\qquad\qquad 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), \qquad 0 \le S \le S_f(T), \\ B_c\left(S_f(t), t\right) = nS_f(t), \qquad 0 \le t \le T, \\ \dfrac{\partial B_c}{\partial S}\left(S_f(t), t\right) = n, \qquad 0 \le t \le T, \\ S_f(T) = \max\left(\dfrac{Z}{n}, \ \dfrac{kZ}{D_0 n}\right). \end{cases}$$

a) Convert this problem into a problem whose solution has a continuous derivative everywhere, and which is defined on a rectangular domain and has an initial condition.
b) Design a pseudo-spectral method to solve the new problem. (Do not require to discuss the solution of the nonlinear system.)

12. Consider the nonlinear system consisting of the following equations:

$$\begin{aligned} &\frac{u_{m,l}^{n+1} - u_{m,l}^n}{\Delta\tau} \\ &= \frac{1}{2}\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}}\left(u_{m,l}^{n+1} + u_{m,l}^n\right) \\ &\quad + \left(\frac{1}{s_{f,l}^{n+1} + s_{f,l}^n}\frac{s_{f,l}^{n+1} - s_{f,l}^n}{\Delta\tau}\right)\xi_m \mathbf{D}_{\xi,\mathbf{m}}\left(u_{m,l}^{n+1} + u_{m,l}^n\right) + a_{7,m,l}, \\ &\qquad\qquad m = 0, 1, \cdots, M-1, \quad l = 0, 1, \cdots, L, \\ &\qquad\qquad u_{M,l}^{n+1} = s_{f,l}^{n+1}, \qquad l = 0, 1, \cdots, L, \end{aligned}$$

and

$$\mathbf{D}_{\xi,\mathbf{M}} u_{M,l}^{n+1} = s_{f,l}^{n+1}, \quad l = 0, 1, \cdots, L,$$

where

$$\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}} = \frac{1}{2}\left(\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1}} + \mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}}\right).$$

Here, $u_{m,l}^n, m = 0, 1, \cdots, M,\ l = 0, 1, \cdots, L$ and $s_{f,l}^n, l = 0, 1, \cdots, L$ are given and $u_{m,l}^{n+1}, m = 0, 1, \cdots, M,\ l = 0, 1, \cdots, L$ and $s_{f,l}^{n+1}, l = 0, 1, \cdots, L$ are unknown. In the system, $\mathbf{D}_{\xi,\mathbf{m}}$ and $\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}}$ are difference operators with variable coefficients. $\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1}}$ is another difference operator whose coefficients depend on $s_{f,l}^{n+1}$, $l = 0, 1, \cdots, L$. Discuss how to solve this system and give an outline of a method that you think is simple and effective.

Projects

General Requirements

A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the student, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

C) If not specified, for each case two results are required. For the first result, a 50×10 mesh should be used. For the second result, the accuracy required is 0.001, and the mesh used should be as coarse as possible.

D) Submit results in form of tables. When a result is given, always provide the problem parameters and the computational parameters.

1. **Implicit scheme (7.22)–(7.24)** Suppose σ, r, D_0 are constant. Write a code performing implicit singularity-separating method for American calls and puts. In the code, a result of an American call option should be obtained by the implicit scheme (7.22)–(7.24), whereas a result of an American put option should be obtained through solving a corresponding call problem numerically and then using the symmetry relation.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$.

- For American call and put options, find the results with an accuracy of 0.00001 under the help of the extrapolation technique. The problem parameters are $S = 90, 100, 110$, $E = 100$, $T = 1.00$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.2$.

2. Using the binomial method (6.28) with (6.25)–(6.27), try to find the values of American call and put options with an accuracy of 0.00001. The problem parameters are $S = 90, 100, 110$, $E = 100$, $T = 1.00$, $r = 0.10$, $D_0 = 0.05$, and $\sigma = 0.2$.

8

Interest Rate Modelling

Problems

1. Define
$$\mathbf{L}_r = \frac{\partial}{\partial r}\left[f_1(r,t)\frac{\partial}{\partial r}\right] - f_2(r,t)\frac{\partial}{\partial r} + f_3(r,t).$$

   a) Find an operator $\mathbf{L}_r^*$ such that
$$\int_{r_l}^{r_u} \mathbf{L}_r V\, U dr = \int_{r_l}^{r_u} \mathbf{L}_r^* U\, V dr + \left[f_1\left(U\frac{\partial V}{\partial r} - V\frac{\partial U}{\partial r}\right) - f_2 VU\right]\Bigg|_{r_l}^{r_u}.$$
   This operator is called the conjugate operator of $\mathbf{L}_r$.

   b) Suppose
$$\frac{\partial V}{\partial t} = -\mathbf{L}_r V, \quad \frac{\partial U}{\partial t} = \mathbf{L}_r^* U,$$
$$f_1(r_l,t) = f_1(r_u,t) = 0, \quad f_2(r_l,t) < 0, \quad f_2(r_u,t) > 0,$$
   and
$$U(r_l,t) = U(r_u,t) = 0.$$
   Show
$$\int_{r_l}^{r_u} U(r,t)V(r,t)dr = constant.$$

   c) Let $U(r,0) = \delta(r - r^*)$ and $V(r,T^*) = 1$. Prove that there is the following relation:
$$V(r^*,0) = \int_{r_l}^{r_u} U(r,T^*)dr.$$

2. Consider the following problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial}{\partial r}\left[f_1(r,t)\dfrac{\partial U}{\partial r}\right] + \dfrac{\partial}{\partial r}\left[f_2\left(r,t,\lambda(t)\sqrt{f_1}\right)U\right] + f_3(r,t)U, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad 0 \le t, \\ U(r,0) = \delta(r - r^*), \qquad\qquad\qquad\qquad r_l \le r \le r_u, \\ U(r_l,t) = 0, \qquad\qquad\qquad\qquad\qquad\quad 0 \le t, \\ U(r_u,t) = 0, \qquad\qquad\qquad\qquad\qquad\quad 0 \le t, \end{cases}$$

where

$$f_1(r,t) \ge 0 \quad \text{and} \quad f_1(r_l,t) = f_1(r_u,t) = 0,$$

and

$$f_2\left(r_l,t,\lambda(t)\sqrt{f_1(r_l,t)}\right) < 0, \quad f_2\left(r_u,t,\lambda(t)\sqrt{f_1(r_u,t)}\right) > 0.$$

Here, $\lambda(t)$ is a unknown function with a known $\lambda(0)$. We want to find such a function $\lambda(t)$ that

$$\int_{r_l}^{r_u} U(r,t)dr = f(t) \qquad \text{for any } t \in [0, T^*_{max}],$$

where $f(t)$ is a given function with $f(0) = 1$. Design a second-order numerical method for this purpose.

3. Design a numerical method for finding the market price of risk by using the bond equation directly and taking the prices of today's zero-coupon bonds with various maturities as input.
4. Design an implicit second-order accurate finite-difference method based on the bond equation to solve the European bond option problem.
5. Design an explicit first-order accurate finite-difference method based on the bond equation to solve a cap problem.
6. What is the difference between the numerical methods for a cap problem and for a floor problem if the bond equation is adopted.
7. Design an implicit second-order accurate finite-difference method based on the bond equation to solve the European swaption problem.
8. What is the difference between the numerical methods for the European swaption problem and for the American swaption problem formulated as a linear complementarity problem if the bond equation is adopted.
9. Assume that the prices of American swaptions are the solutions of the following linear complementarity problem:

$$\begin{cases} \min\Bigg(-\dfrac{\partial V_{so}}{\partial t} - \dfrac{1}{2}w^2\dfrac{\partial^2 V_{so}}{\partial r^2} - (u - \lambda w)\dfrac{\partial V_{so}}{\partial r} + rV_{so}, \\ \qquad\qquad V_{so}(r,t) - \max\left(V_s(r,t;r_{se},t),0\right)\Bigg) = 0, \\ V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T),0\right), \end{cases}$$

where $t \in [0, T]$ and $r \in [r_l, r_u]$ and $V_s(r, t; r_{se}, t)$ is the price of the swap. Suppose that the price of the swap has been found and assume that there is only one free boundary. Formulate this problem as a free-boundary problem.

10. Suppose that the free-boundary problem of the American swaption problem based on the bond equation is given by

$$\begin{cases} \dfrac{\partial V_{so}}{\partial t} + \dfrac{1}{2} w^2 \dfrac{\partial^2 V_{so}}{\partial r^2} + (u - \lambda w) \dfrac{\partial V_{so}}{\partial r} - r V_{so} = 0, & \\ & r_l \le r \le r_f(t), \quad t \le T, \\ V_{so}(r, t) = \max\left(V_s(r, T; r_{se}, T), 0\right), & r_l \le r \le r_f(T), \\ V_{so}(r_f(t), t) = V_s(r_f(t), t; r_{se}, t), & t \le T, \\ \dfrac{\partial V_{so}(r_f(t), t)}{\partial r} = \dfrac{\partial V_s(r_f(t), t; r_{se}, t)}{\partial r}, & t \le T \\ r_f(T) = \max(r_0^*, r_1^*), & \end{cases}$$

where r_0^* satisfies

$$V_s(r_0^*, T; S_{se}, T) = 0$$

and r_1^* is the point between the interval where

$$\frac{\partial V_s}{\partial t} + \frac{1}{2} w^2 \frac{\partial V_s}{\partial r^2} + (u - \lambda w) \frac{\partial V_s}{\partial r} - r V_s \ge 0$$

and the interval where

$$\frac{\partial V_s}{\partial t} + \frac{1}{2} w^2 \frac{\partial V_s}{\partial r^2} + (u - \lambda w) \frac{\partial V_s}{\partial r} - r V_s < 0.$$

Here V_s denotes $V_s(r, T; r_{se}, T)$. Briefly describe how to solve this problem by an implicit second-order finite-difference method.

11. Briefly describe how to solve a European swaption problem numerically by using the three-factor interest rate model.
12. Design a method for determining the value of a bond option by using a two-factor or three-factor interest rate model.
13. Design a method for evaluating the price of a cap by using the three-factor interest rate model.

Projects

General Requirements

A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.
B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the student, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.
C) For each case, two results are required. One result is on a 20×24 mesh, and the accuracy of the other result will be specified individually. (The error of the solution on a 20×24 mesh might be quite large.)
D) Submit results in form of tables. When a result is given, always provide the problem parameters and the computational parameters.

1. **Implicit method (5.31) with modification at the boundaries for European bond options and swaptions**. Suppose

$$dr = (0.05345 - r)dt + r(0.2 - r)dX, \quad r_l = 0 \le r \le r_u = 0.2$$

and $\lambda(t)$ has been found and is given as a function in C. Also, assume that today's spot interest rate is 0.05345. Write a code to calculate European bond options and a code to calculate European swaptions.

- For European bond options, give results for the case: $E = 0.95, 1$, $k = 0.055$, $T = 0.25, 0.5$, and $T_b - T = 1, 2$. The requirement on the accuracy of the other result is 0.0001, and the mesh used should be as coarse as possible.
- For swaptions, give the results for the cases: $Q = 100$, $N = 5, 10$, $T = 0.5, 1, 2$, $r_{se} = 0.05507$ for $N = 5$, and $r_{se} = 0.05766$ for $N = 10$. The requirement on the accuracy of the other result is 0.001, and the mesh used should be as coarse as possible.