5

Basic Numerical Methods

Problems

1. Suppose $x_m = m\Delta x$.
   a) Find the order of the error of the following approximate function

$$u(x) \approx \frac{x_{m+1} - x}{\Delta x} u(x_m) + \frac{x - x_m}{\Delta x} u(x_{m+1})$$

   by the Taylor expansion. Here $x \in [x_m, x_{m+1}]$.

   b) Find the order of the error of the following approximate function

$$\begin{aligned} u(x) \approx & \frac{(x - x_m)(x - x_{m+1})}{2\Delta x^2} u(x_{m-1}) \\ & - \frac{(x - x_{m-1})(x - x_{m+1})}{\Delta x^2} u(x_m) \\ & + \frac{(x - x_{m-1})(x - x_m)}{2\Delta x^2} u(x_{m+1}) \end{aligned}$$

   by the Taylor expansion. Here $x \in [x_{m-1}, x_{m+1}]$.

Solution:

a) The Taylor series gives

$$\begin{cases} u\left(x_m\right) = u\left(x\right) + u'\left(x\right)\left(x_m - x\right) + \dfrac{u''\left(\xi_1\right)}{2!}\left(x_m - x\right)^2, \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi_1 \in \left[x_m, x\right], \\ u\left(x_{m+1}\right) = u\left(x\right) + u'\left(x\right)\left(x_{m+1} - x\right) + \dfrac{u''\left(\xi_2\right)}{2!}\left(x_{m+1} - x\right)^2, \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi_2 \in \left[x, x_{m+1}\right]. \end{cases}$$

From there we have

$$\begin{aligned}
&\frac{x_{m+1}-x}{\Delta x}u\left(x_m\right)+\frac{x-x_m}{\Delta x}u\left(x_{m+1}\right)\\
=&\frac{u\left(x\right)}{\Delta x}\left(x_{m+1}-x+x-x_m\right)\\
&+\frac{u'\left(x\right)}{\Delta x}\left[\left(x_{m+1}-x\right)\left(x_m-x\right)+\left(x-x_m\right)\left(x_{m+1}-x\right)\right]\\
&+\frac{\left(x_{m+1}-x\right)\left(x_m-x\right)}{2!\Delta x}\left[u''\left(\xi_1\right)\left(x_m-x\right)-u''\left(\xi_2\right)\left(x_{m+1}-x\right)\right]\\
=&\,u\left(x\right)+\frac{\left(x_{m+1}-x\right)\left(x_m-x\right)}{2!\Delta x}\left[u''\left(\xi_1\right)\left(x_m-x\right)-u''\left(\xi_2\right)\left(x_{m+1}-x\right)\right].
\end{aligned}$$

Therefore the order of the error is $O(\Delta x^2)$.

b) With the Taylor expansion, we have the following for x_m, x_{m-1}, x_{m+1}

$$\begin{aligned}
u(x_{m-1}) &= u(x)+(x_{m-1}-x)\frac{\partial u(x)}{\partial x}+\frac{1}{2}(x_{m-1}-x)^2\frac{\partial^2 u(x)}{\partial x^2}\\
&\quad+O(\Delta x^3),\\
u(x_m) &= u(x)+(x_m-x)\frac{\partial u(x)}{\partial x}+\frac{1}{2}(x_m-x)^2\frac{\partial^2 u(x)}{\partial x^2}\\
&\quad+O(\Delta x^3),\\
u(x_{m+1}) &= u(x)+(x_{m+1}-x)\frac{\partial u(x)}{\partial x}+\frac{1}{2}(x_{m+1}-x)^2\frac{\partial^2 u(x)}{\partial x^2}\\
&\quad+O(\Delta x^3).
\end{aligned}$$

These relations can be rewritten as

$$\begin{aligned}
&\begin{bmatrix} 1 & x_{m-1}-x & (x_{m-1}-x)^2 \\ 1 & x_m-x & (x_m-x)^2 \\ 1 & x_{m+1}-x & (x_{m+1}-x)^2 \end{bmatrix}\begin{bmatrix} u(x) \\ \frac{\partial u(x)}{\partial x} \\ \frac{1}{2}\frac{\partial^2 u(x)}{\partial x^2} \end{bmatrix}\\
=&\begin{bmatrix} u(x_{m-1})+O(\Delta x^3) \\ u(x_m)+O(\Delta x^3) \\ u(x_{m+1})+O(\Delta x^3) \end{bmatrix}.
\end{aligned}$$

Because

$$\begin{aligned}
&\begin{vmatrix} 1 & x_{m-1}-x & (x_{m-1}-x)^2 \\ 1 & x_m-x & (x_m-x)^2 \\ 1 & x_{m+1}-x & (x_{m+1}-x)^2 \end{vmatrix}\\
=&\begin{vmatrix} 1 & x_{m-1}-x & (x_{m-1}-x)^2 \\ 0 & x_m-x_{m-1} & (x_m-x)^2-(x_{m-1}-x)^2 \\ 0 & x_{m+1}-x_{m-1} & (x_{m+1}-x)^2-(x_{m-1}-x)^2 \end{vmatrix}\\
=&\,(x_m-x_{m-1})(x_{m+1}-x_{m-1})\left[x_{m+1}+x_{m-1}-2x\right.\\
&\left.-(x_m+x_{m-1}-2x)\right]\\
=&\,(x_m-x_{m-1})(x_{m+1}-x_{m-1})(x_{m+1}+x_m),
\end{aligned}$$

we have

$$
\begin{aligned}
&u(x)\\
&=\frac{\begin{vmatrix} u_{m-1}+O(\Delta x^3) & x_{m-1}-x & (x_{m-1}-x)^2\\ u_m+O(\Delta x^3) & x_m-x & (x_m-x)^2\\ u_{m+1}+O(\Delta x^3) & x_{m+1}-x & (x_{m+1}-x)^2 \end{vmatrix}}{(x_m-x_{m-1})(x_{m+1}-x_{m-1})(x_{m+1}-x_m)}\\
&=(u_{m-1}+O(\Delta x^3))\frac{(x_m-x)(x_{m+1}-x)(x_{m+1}-x-x_m+x)}{(x_m-x_{m-1})(x_{m+1}-x_{m-1})(x_{m+1}-x_m)}\\
&\quad-(u_m+O(\Delta x^3))\frac{(x_{m-1}-x)(x_{m+1}-x)(x_{m+1}-x-x_{m-1}+x)}{(x_m-x_{m-1})(x_{m+1}-x_{m-1})(x_{m+1}-x_m)}\\
&\quad+(u_{m+1}+O(\Delta x^3))\frac{(x_{m-1}-x)(x_m-x)(x_m-x-x_{m-1}+x)}{(x_m-x_{m-1})(x_{m+1}-x_{m-1})(x_{m+1}-x_m)}\\
&=u_{m-1}\frac{(x_m-x)(x_{m+1}-x)}{(x_m-x_{m-1})(x_{m+1}-x_{m-1})}+u_m\frac{(x_{m-1}-x)(x_{m+1}-x)}{(x_{m-1}-x_m)\,(x_{m+1}-x_m)}\\
&\quad+u_{m+1}\frac{(x_{m-1}-x)(x_m-x)}{(x_{m-1}-x_{m+1})(x_m-x_{m+1})}+O(\Delta x^3).
\end{aligned}
$$

Consequently, the order of the error is $O(\Delta x^3)$.

2. Show that from

$$
\begin{cases}
a_{m+1}=a_m+b_mh_m+c_mh_m^2+d_mh_m^3,\\
b_{m+1}=b_m+2c_mh_m+3d_mh_m^2,\\
c_{m+1}=c_m+3d_mh_m,\\
\qquad m=0,1,\cdots,M-2,
\end{cases}
$$

and

$$
\begin{cases}
a_M=a_{M-1}+b_{M-1}h_{M-1}+c_{M-1}h_{M-1}^2+d_{M-1}h_{M-1}^3,\\
c_M=c_{M-1}+3d_{M-1}h_{M-1},
\end{cases}
$$

the following relation can be derived:

$$
\begin{aligned}
&h_{m-1}c_{m-1}+2(h_{m-1}+h_m)c_m+h_mc_{m+1}\\
&=\frac{3(a_{m+1}-a_m)}{h_m}-\frac{3(a_m-a_{m-1})}{h_{m-1}},\\
&\qquad m=1,2,\cdots,M-1.
\end{aligned}
$$

Solution:
First let us eliminate b_m. From the given equations, we have

$$3\,(a_{m+1}-a_m)=3b_mh_m+3c_mh_m^2+3d_mh_m^3,\quad m=0,1,\cdots,M-1,$$

which can be further rewritten as

$$\frac{3(a_{m+1}-a_m)}{h_m}-\frac{3(a_m-a_{m-1})}{h_{m-1}}$$
$$=3(b_m-b_{m-1})+3c_mh_m-3c_{m-1}h_{m-1}+3d_mh_m^2-3d_{m-1}h_{m-1}^2,$$
$$m=1,2,\cdots,M-1.$$

From the given equations, we also have

$$3(b_m-b_{m-1})=6c_{m-1}h_{m-1}+9d_{m-1}h_{m-1}^2,\quad m=1,2,\cdots,M-1.$$

Therefore

$$\frac{3(a_{m+1}-a_m)}{h_m}-\frac{3(a_m-a_{m-1})}{h_{m-1}}$$
$$=3c_mh_m+3c_{m-1}h_{m-1}+3d_mh_m^2+6d_{m-1}h_{m-1}^2,$$

where $m=1,2,\cdots,M-1$. Now let us eliminate d_m. Those relations among c_m, c_{m-1} and d_{m-1} can be written as

$$c_mh_{m-1}=c_{m-1}h_{m-1}+3d_{m-1}h_{m-1}^2,\quad m=1,2,\cdots,M-1.$$

Consequently we arrive at

$$\frac{3(a_{m+1}-a_m)}{h_m}-\frac{3(a_m-a_{m-1})}{h_{m-1}}$$
$$=3c_mh_m+3c_{m-1}h_{m-1}+c_{m+1}h_m-c_mh_m+2c_mh_{m-1}-2c_{m-1}h_{m-1}$$
$$=h_{m-1}c_{m-1}+2(h_{m-1}+h_m)c_m+h_mc_{m+1},$$

where $m=1,2,\cdots,M-1$.

3. Consider the cubic spline problem. Suppose that the derivative is given at $x=x_M$, insteady of assuming $c_M=0$. Derive the equation which should replace the equation $c_M=0$ in the system for $c_0,c_1,\cdots,c_M$.
Solution: If the derivative $f'(x_M)$ is given, then at $x=x_M$ we have

$$\begin{cases} a_M=a_{M-1}+b_{M-1}h_{M-1}+c_{M-1}h_{M-1}^2+d_{M-1}h_{M-1}^3, \\ f'(x_M)=b_{M-1}+2c_{M-1}h_{M-1}+3d_{M-1}h_{M-1}^2, \\ c_M=c_{M-1}+3d_{M-1}h_{M-1}. \end{cases}$$

Solving the third equation for d_{M-1} and eliminating d_{M-1} from the first and second equations, we have

$$\begin{cases} \dfrac{a_M-a_{M-1}}{h_{M-1}}=b_{M-1}+c_{M-1}h_{M-1}+\dfrac{c_M-c_{M-1}}{3}h_{M-1}, \\ f'(x_M)=b_{M-1}+2c_{M-1}h_{M-1}+(c_M-c_{M-1})h_{M-1}. \end{cases}$$

Eliminating b_{M-1} from the two equations yields

$$f'(x_M) - \frac{a_M - a_{M-1}}{h_{M-1}} = h_{M-1}c_{M-1} + \frac{2}{3}(c_M - c_{M-1})h_{M-1}$$
$$= \frac{1}{3}h_{M-1}c_{M-1} + \frac{2}{3}h_{M-1}c_M.$$

This relation can be rewritten as

$$h_{M-1}c_{M-1} + 2h_{M-1}c_M = 3\left(f'(x_M) - \frac{a_M - a_{M-1}}{h_{M-1}}\right),$$

which should replace the equation $c_M = 0$ in the system for $c_0, c_1, \cdots, c_M$.

4. Suppose $x_m = m\Delta x$, $y_l = l\Delta y$, and $\tau^n = n\Delta\tau$. Find the expression of the error of each of the following approximations:
   a) $u(x_m, \tau^{n+1/2}) \approx \dfrac{u(x_m, \tau^{n+1}) + u(x_m, \tau^n)}{2}$;
   b) $\dfrac{\partial u}{\partial \tau}(x_m, \tau^n) \approx \dfrac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}$;
   c) $\dfrac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) \approx \dfrac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}$;
   d) $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - u(x_m, \tau^n)}{\Delta x}$;
   e) $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - u(x_{m-1}, \tau^n)}{2\Delta x}$;
   f) $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{3u(x_m, \tau^n) - 4u(x_{m-1}, \tau^n) + u(x_{m-2}, \tau^n)}{2\Delta x}$;
   g) $\dfrac{\partial^2 u}{\partial x^2}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - 2u(x_m, \tau^n) + u(x_{m-1}, \tau^n)}{\Delta x^2}$;
   h)

$$\frac{\partial^2 u}{\partial x \partial y}(x_m, y_l, \tau^n) \approx \frac{1}{2\Delta x}\left[\frac{u(x_{m+1}, y_{l+1}, \tau^n) - u(x_{m+1}, y_{l-1}, \tau^n)}{2\Delta y} - \frac{u(x_{m-1}, y_{l+1}, \tau^n) - u(x_{m-1}, y_{l-1}, \tau^n)}{2\Delta y}\right].$$

Solution:

a) For $u(x_m, \tau^{n+1})$ and $u(x_m, \tau^n)$, we have

$$u(x_m, \tau^{n+1}) = u(x_m, \tau^{n+1/2}) + \frac{\Delta\tau}{2} \cdot \frac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2 \frac{\partial^2 u(x_m, \eta_1)}{\partial \tau^2},$$

$$u(x_m, \tau^n) = u(x_m, \tau^{n+1/2}) - \frac{\Delta\tau}{2} \cdot \frac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2 \frac{\partial^2 u(x_m, \eta_2)}{\partial \tau^2},$$

where $\tau^{n+1/2} \le \eta_1 \le \tau^{n+1}$ and $\tau^n \le \eta_2 \le \tau^{n+1/2}$. Adding the above two equations together and dividing by 2, we get the following:

$$u(x_m,\tau^{n+1/2}) = \frac{u(x_m,\tau^{n+1}) + u(x_m,\tau^n)}{2} - \frac{\Delta\tau^2}{8}\frac{\partial^2 u}{\partial \tau^2}(x_m,\eta_3),$$

where $\tau^n \le \eta_3 \le \tau^{n+1}$. Therefore, the truncation error is $O(\Delta\tau^2)$.

b)

$$u(x_m,\tau^{n+1}) = u(x_m,\tau^n) + \Delta\tau\frac{\partial u(x_m,\tau^n)}{\partial\tau} + \frac{\Delta\tau^2}{2}\frac{\partial^2 u(x_m,\eta)}{\partial\tau^2},$$

i.e.,

$$\frac{\partial u(x_m,\tau^n)}{\partial\tau} = \frac{u(x_m,\tau^{n+1}) - u(x_m,\tau^n)}{\Delta\tau} - \frac{\Delta\tau}{2}\cdot\frac{\partial^2 u(x_m,\eta)}{\partial\tau^2},$$

$$\tau^n \le \eta \le \tau^{n+1}.$$

Therefore, the truncation error is $O(\Delta\tau)$.

c) Because

$$\begin{aligned}
&u(x_m,\tau^{n+1})\\
&= u(x_m,\tau^{n+1/2}) + \frac{\Delta\tau}{2}\cdot\frac{\partial u}{\partial\tau}(x_m,\tau^{n+1/2})\\
&\quad + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2\frac{\partial^2 u(x_m,\tau^{n+1/2})}{\partial\tau^2} + \frac{1}{6}\cdot\left(\frac{\Delta\tau}{2}\right)^3\frac{\partial^3 u}{\partial\tau^3}(x_m,\eta_1),\\
&u(x_m,\tau^n)\\
&= u(x_m,\tau^{n+1/2}) - \frac{\Delta\tau}{2}\cdot\frac{\partial u}{\partial\tau}(x_m,\tau^{n+1/2})\\
&\quad + \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2\frac{\partial^2 u(x_m,\tau^{n+1/2})}{\partial\tau^2} - \frac{1}{6}\cdot\left(\frac{\Delta\tau}{2}\right)^3\frac{\partial^3 u}{\partial\tau^3}(x_m,\eta_2),
\end{aligned}$$

we have

$$\frac{\partial u(x_m,\tau^{n+1/2})}{\partial\tau} = \frac{u(x_m,\tau^{n+1}) - u(x_m,\tau^n)}{\Delta\tau} - \frac{\Delta\tau^2}{24}\frac{\partial^3 u(x_m,\eta_3)}{\partial\tau^3}.$$

Therefore, the truncation error is $O(\Delta\tau^2)$, and η_1,η_2 and η_3 is as defined in a).

d)

$$u(x_{m+1},\tau^n) = u(x_m,\tau^n) + \Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u(\xi,\tau^n)}{\partial x^2},$$

$$\frac{\partial u(x_m,\tau^n)}{\partial x} = \frac{u(x_{m+1},\tau^n) - u(x_m,\tau^n)}{\Delta x} - \frac{\Delta x}{2}\frac{\partial^2 u(\xi,\tau^n)}{\partial x^2}.$$

Therefore, the truncation error is $O(\Delta x)$ and $x_m \le \xi \le x_{m+1}$.

e)

$$u(x_{m+1},\tau^n) = u(x_m,\tau^n) + \Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u(\xi_1,\tau^n)}{\partial x^3},$$

$$u(x_{m-1},\tau^n) = u(x_m,\tau^n) - \Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u(\xi_2,\tau^n)}{\partial x^3},$$

where $x_m \le \xi_1 \le x_{m+1}$ and $x_{m-1} \le \xi_2 \le x_m$. Subtracting the second from the first equation and dividing by $2\Delta x$ yield

$$\frac{\partial u(x_m,\tau^n)}{\partial x} = \frac{u(x_{m+1},\tau^n) - u(x_{m-1},\tau^n)}{2\Delta x} - \frac{\Delta x^2}{6}\frac{\partial^3 u(\xi_3,\tau^n)}{\partial x^3},$$

where $x_{m-1} \le \xi_3 \le x_{m+1}$. Therefore, the truncation error is $O(\Delta x^2)$

f)

$$u(x_{m-1},\tau^n) = u(x_m,\tau^n) - \Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{1}{2}\Delta x^2\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2} - \frac{1}{6}\Delta x^3\frac{\partial^3 u(\xi_1,\tau^n)}{\partial x^3},$$

$$u(x_{m-2},\tau^n) = u(x_m,\tau^n) - 2\Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{4}{2}\Delta x^2\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2} - \frac{8}{6}\Delta x^3\frac{\partial^3 u(\xi_2,\tau^n)}{\partial x^3},$$

where $x_{m-1} \le \xi_1 \le x_m$ and $x_{m-2} \le \xi_2 \le x_m$. Multiplying the first equation by four and subtracting the second equation yield

$$\frac{\partial u(x_m,\tau^n)}{\partial x} = \frac{3u(x_m,\tau^n) - 4u(x_{m-1},\tau^n) + u(x_{m-2},\tau^n)}{2\Delta x} + \frac{1}{3}\Delta x^2\left(2\frac{\partial^3 u(\xi_2,\tau^n)}{\partial x^3} - \frac{\partial^3 u(\xi_1,\tau^n)}{\partial x^3}\right).$$

Therefore, the truncation error is $O(\Delta x^2)$.

g)

$$u(x_{m+1},\tau^n) = u(x_m,\tau^n) + \Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u(x_m,\tau^n)}{\partial x^3} + \frac{\Delta x^4}{24}\frac{\partial^4 u(\xi_1,\tau^n)}{\partial x^4},$$

$$u(x_{m-1},\tau^n) = u(x_m,\tau^n) - \Delta x\frac{\partial u(x_m,\tau^n)}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u(x_m,\tau^n)}{\partial x^3} + \frac{\Delta x^4}{24}\frac{\partial^4 u(\xi_2,\tau^n)}{\partial x^4},$$

where $x_m \le \xi_1 \le x_{m+1}$ and $x_{m-1} \le \xi_2 \le x_m$. Add the above two equations together, we have

$$\begin{aligned}&\frac{\partial^2 u(x_m,\tau^n)}{\partial x^2}\\ &= \frac{u(x_{m+1},\tau^n)-2u(x_m,\tau^n)+u(x_{m-1},\tau^n)}{\Delta x^2}-\frac{\Delta x^2}{12}\frac{\partial^4 u(\xi_3,\tau^n)}{\partial x^4},\end{aligned}$$

where $x_{m-1} \le \xi_3 \le x_{m+1}$. Therefore, the truncation error is $O(\Delta x^2)$.

h) Similar to e), we have

$$\begin{aligned}&\frac{\partial u}{\partial y}(x_m,y_l,\tau^n)\\ &= \frac{u(x_m,y_{l+1},\tau^n)-u(x_m,y_{l-1},\tau^n)}{2\Delta y}-\frac{\Delta y^2}{6}\frac{\partial^3 u}{\partial y^3}(x_m,\eta,\tau^n).\end{aligned}$$

Using the results in e), we can further have

$$\begin{aligned}&\frac{\partial^2 u}{\partial x\partial y}(x_m,y_l,\tau^n)\\ &= \left[\frac{u(x_{m+1},y_{l+1},\tau^n)-u(x_{m+1},y_{l-1},\tau^n)}{2\Delta y}\right.\\ &\quad -\frac{\Delta y^2}{6}\frac{\partial^3 u}{\partial y^3}(x_{m+1},\eta_1,\tau^n)\\ &\quad -\frac{u(x_{m-1},y_{l+1},\tau^n)-u(x_{m-1},y_{l-1},\tau^n)}{2\Delta y}\\ &\quad \left.+\frac{\Delta y^2}{6}\frac{\partial^3 u}{\partial y^3}(x_{m-1},\eta_2,\tau^n)\right]\frac{1}{2\Delta x}\\ &\quad -\frac{\Delta x^2}{6}\frac{\partial^4 u}{\partial x^3\partial y}(\xi_1,y_l,\tau^n)\\ &= \frac{1}{2\Delta x}\left[\frac{u(x_{m+1},y_{l+1},\tau^n)-u(x_{m+1},y_{l-1},\tau^n)}{2\Delta y}\right.\\ &\quad \left.-\frac{u(x_{m-1},y_{l+1},\tau^n)-u(x_{m-1},y_{l-1},\tau^n)}{2\Delta y}\right]+O(\Delta x^2)+O(\Delta y^2).\end{aligned}$$

Therefore the truncation error is $O(\Delta x^2)+O(\Delta y^2)$, where we assume $\Delta y = O(\Delta x)$.

5. For $y \in [-1,1]$, we define

$$T_N(y) = \cos\left(N\cos^{-1} y\right),$$

where N is an integer. Let

$$y_j = \cos\frac{j\pi}{N}, \quad j = 0,1,\cdots,N.$$

Show

a) $T_{k+1}(y) - 2yT_k(y) + T_{k-1}(y) = 0,\ k \geq 1.$
b) $T_N(y)$ is a polynomial of degree N for any nonnegative integer.
c) $$\frac{dT_N(y_j)}{dy} = \begin{cases} N^2, & j = 0, \\ 0, & j = 1, 2, \cdots, N-1, \\ (-1)^{N+1} N^2, & j = N; \end{cases}$$
d) $$\frac{d^2T_N(y_j)}{dy^2} = \begin{cases} \dfrac{N^2\left(N^2-1\right)}{3}, & j = 0, \\ \dfrac{(-1)^{j+1} N^2}{\left(1-y_j^2\right)}, & j = 1, 2, \cdots, N-1, \\ \dfrac{(-1)^{N} N^2\left(N^2-1\right)}{3}, & j = N; \end{cases}$$
e) $$\frac{d^3T_N(y_j)}{dy^3} = \frac{(-1)^{j+1} 3N^2 y_j}{\left(1-y_j^2\right)^2},\ j = 1, 2, \cdots, N-1.$$

Solution:

a) From $\cos((k \pm 1)z) = \cos kz \cos z \mp \sin kz \sin z$, we have $\cos(k+1)z + \cos(k-1)z = 2\cos kz \cos z$. Let $\cos z = y$, i.e., $z = \cos^{-1} y$, we arrive at
$$T_{k+1}(y) - 2yT_k(y) + T_{k-1}(y) = 0, \quad k \geq 1.$$
b) Suppose that $T_k(y)$ is a polynomial with degree k, $k = 0, 1, \cdots, N-1$. Because $T_N(y) = 2yT_{N-1}(y) - T_{N-2}(y)$, $T_N(y)$ must be a polynomial with degree N. We know
$$\begin{aligned} T_0(y) &= 1 \ \ \text{(degree 0)}, \\ T_1(y) &= y \ \ \text{(degree 1)}. \end{aligned}$$
Thus, by the induction method, we know that $T_k(y)$ is a polynomial with degree $k, k = 2, 3, \cdots$.
c) Let $z = \cos^{-1} y$. Then $T_N(y) = T_N(\cos z) = \cos Nz$ and $dy = -\sin z dz$. Thus
$$\frac{dT_N(y)}{dy} = \frac{d\cos Nz}{dz}\frac{dz}{dy} = \frac{-N\sin Nz}{-\sin z} = \frac{N\sin Nz}{\sin z}.$$
Consequently, we have
$$\frac{dT_N(y_j)}{dy} = \frac{N\sin j\pi}{\sin\frac{j\pi}{N}} = \begin{cases} \lim\limits_{j\to 0}\dfrac{N\cos j\pi \cdot \pi}{\cos\frac{j\pi}{N} \cdot \frac{\pi}{N}} = N^2, & j = 0, \\ 0, & j = 1, 2, \cdots, N-1, \\ \lim\limits_{j\to N}\dfrac{N\cos j\pi \cdot \pi}{\cos\frac{j\pi}{N} \cdot \frac{\pi}{N}} = (-1)^{N+1} N^2, & j = N. \end{cases}$$