we do not specify any value for $U(r_l,t)$ as a boundary condition. For the upper boundary, the situation is similar. Under such a treatment, (8.6) and (8.8) still hold.

8.1.2 Numerical Methods for the Inverse Problem

Again, let $\overline{V}(T^*)$ denote today's zero-coupon bond curve for bonds with a face value $Z = 1$. Suppose that the values of K zero-coupon bonds with maturities $T_1^*, T_2^*, \cdots, T_K^*$ are $V_{T_1^*}, V_{T_2^*}, \cdots, V_{T_K^*}$, which can be obtained from the market. Assume $T_K^* = T_{max}^*$ and $0 < T_1^* < \cdots < T_K^*$. Let today's time be T_0^* and $T_0^* = 0$. Clearly, $\overline{V}(T_0^*) = 1$ and $\dfrac{\partial \overline{V}(T_0^*)}{\partial T^*} = -r^*$, where r^* is today's spot interest rate. Based on the data, we can generate a zero-coupon bond price curve $\overline{V}(T^*)$ on $[0, T_{max}^*]$ by the cubic spline interpolation described in Subsection 5.1.1. Because $\dfrac{\partial \overline{V}(T_0^*)}{\partial T^*} = -r^*$, at the left end we require this condition instead of assuming $\dfrac{\partial^2 \overline{V}(T_0^*)}{\partial T^{*2}} = 0$. At the right end, we assume the function $\overline{V}(T^*)$ to be a polynomial of degree two on $[T_{K-1}^*, T_K^*]$ instead of assuming $\dfrac{\partial^2 \overline{V}(T_M^*)}{\partial T^{*2}} = 0$. Using the method described in Subsection 5.1.1 for the modified case, we can determine these polynomials on all the subintervals $[T_k^*, T_{k+1}^*]$, $k = 0, 1, \cdots, K-1$. As soon as we have the zero-coupon bond curve, we can determine $\lambda(t)$ by solving inverse problems.

First, let us discuss how to solve the inverse problem (4.47). When $\lambda(t)$ is given on $[0, T^*]$, the partial differential equation can be discretized by (5.37). Hence, for any T^*, as long as $\lambda(t)$ is given on $[0, T^*]$, we can calculate $V(r, 0; T^*)$ from $V(r, T^*; T^*)$. Assume that we have obtained $\lambda(t)$ on $[0, T^* - \Delta t]$ from the value $\overline{V}(t)$ on $[0, T^* - \Delta t]$. We guess $\lambda(T^*)$, assume $\lambda(t)$ to be a linear function on $[T^* - \Delta t, T^*]$, and solve (4.47) from $t = T^*$ to $t = 0$. Check if $V(r^*, 0; T^*) = \overline{V}(T^*)$. If it is true, we find $\lambda(t)$ on $[T^* - \Delta t, T^*]$; if not, we adjust $\lambda(T^*)$ until we find a value $\lambda(T^*)$ such that $V(r^*, 0; T^*) = \overline{V}(T^*)$. This procedure can start from $T^* = \Delta t$ and continue successively until $T^* = T_{max}^*$. At $T^* = \Delta t$, if only $\lambda(\Delta t)$ is given, we cannot define a linear function on $[0, \Delta t]$. From (8.8), we see that $\lambda(0)$ can be determined by

$$\lambda(0) = \frac{\dfrac{\partial^2 \overline{V}(0)}{\partial T^{*2}} - r^{*2} + u(r^*, 0)}{w(r^*, 0)}. \tag{8.11}$$

Now let us discuss how to solve (8.7). For the domain $[r_l, r_u] \times [0, T_{max}^*]$, we take the following partition: $r_m = r_l + m\Delta r$, $m = 0, 1, \cdots, M$, $t^n = n\Delta t$, $n = 0, 1, \cdots, N$, where $\Delta r = (r_u - r_l)/M$ and $\Delta t = T_{max}^*/N$, M, N being integers. Let U_m^n and $\lambda^{n+1/2}$ be the approximate values of $U(r_m, t^n)$ and $\lambda(t^{n+1/2})$, and $\overline{V}^n$ denote $\overline{V}(t^n)$. We also represent U_m^n, $m = 0, 1, \cdots, M$ by $\{U_m^n\}$. On this partition, (8.7) and (8.8) can be discretized as follows.

Because the initial condition in (8.7) is a Dirac delta function, we discretize the partial differential equation there by the following "conservative" scheme:

$$\begin{aligned}
&\frac{U_m^{n+1}-U_m^n}{\Delta t} \\
&= \frac{1}{4\Delta r}\left[\left(\bar{w}_{m+1/2}^{n+1/2}\right)^2\left(\frac{U_{m+1}^{n+1}-U_m^{n+1}}{\Delta r}+\frac{U_{m+1}^n-U_m^n}{\Delta r}\right)\right. \\
&\qquad\left. -\left(\bar{w}_{m-1/2}^{n+1/2}\right)^2\left(\frac{U_m^{n+1}-U_{m-1}^{n+1}}{\Delta r}+\frac{U_m^n-U_{m-1}^n}{\Delta r}\right)\right] \\
&\quad -\left[\bar{u}_{m+1/2}^{n+1/2}-\left(\lambda^{n+1/2}+\frac{w_{m+1}^{n+1/2}-w_m^{n+1/2}}{\Delta r}\right)\bar{w}_{m+1/2}^{n+1/2}\right] \\
&\qquad\times\frac{U_{m+1}^{n+1}+U_m^{n+1}+U_{m+1}^n+U_m^n}{4\Delta r} \\
&\quad +\left[\bar{u}_{m-1/2}^{n+1/2}-\left(\lambda^{n+1/2}+\frac{w_m^{n+1/2}-w_{m-1}^{n+1/2}}{\Delta r}\right)\bar{w}_{m-1/2}^{n+1/2}\right] \\
&\qquad\times\frac{U_m^{n+1}+U_{m-1}^{n+1}+U_m^n+U_{m-1}^n}{4\Delta r} \\
&\quad -\frac{r_m}{2}\left(U_m^{n+1}+U_m^n\right), \\
&\qquad m=1,2,\cdots,M-1,
\end{aligned} \tag{8.12}$$

where $\bar{w}_{m+1/2}^{n+1/2}=(w_{m+1}^{n+1/2}+w_m^{n+1/2})/2$ and $\bar{u}_{m+1/2}^{n+1/2}=(u_{m+1}^{n+1/2}+u_m^{n+1/2})/2$. This system can be rewritten as

$$a_m U_{m-1}^{n+1}+b_m U_m^{n+1}+c_m U_{m+1}^{n+1}=-a_m U_{m-1}^n+(2-b_m)U_m^n-c_m U_{m+1}^n,$$
$$m=1,2,\cdots,M-1,$$

where

$$\begin{aligned}
a_m &= \frac{-\Delta t}{4\Delta r^2}\left(\bar{w}_{m-1/2}^{n+1/2}\right)^2 \\
&\quad -\frac{\Delta t}{4\Delta r}\left[\bar{u}_{m-1/2}^{n+1/2}-\left(\lambda^{n+1/2}+\frac{w_m^{n+1/2}-w_{m-1}^{n+1/2}}{\Delta r}\right)\bar{w}_{m-1/2}^{n+1/2}\right], \\
b_m &= 1+\frac{\Delta t r_m}{2}+\frac{\Delta t}{4\Delta r^2}\left[\left(\bar{w}_{m+1/2}^{n+1/2}\right)^2+\left(\bar{w}_{m-1/2}^{n+1/2}\right)^2\right] \\
&\quad +\frac{\Delta t}{4\Delta r}\left[\bar{u}_{m+1/2}^{n+1/2}-\left(\lambda^{n+1/2}+\frac{w_{m+1}^{n+1/2}-w_m^{n+1/2}}{\Delta r}\right)\bar{w}_{m+1/2}^{n+1/2}\right] \\
&\quad -\frac{\Delta t}{4\Delta r}\left[\bar{u}_{m-1/2}^{n+1/2}-\left(\lambda^{n+1/2}+\frac{w_m^{n+1/2}-w_{m-1}^{n+1/2}}{\Delta r}\right)\bar{w}_{m-1/2}^{n+1/2}\right],
\end{aligned}$$

$$
\begin{aligned}
c_m = & \frac{-\Delta t}{4\Delta r^2}\left(\bar{w}_{m+1/2}^{n+1/2}\right)^2 \\
& +\frac{\Delta t}{4\Delta r}\left[\bar{u}_{m+1/2}^{n+1/2}-\left(\lambda^{n+1/2}+\frac{w_{m+1}^{n+1/2}-w_m^{n+1/2}}{\Delta r}\right)\bar{w}_{m+1/2}^{n+1/2}\right].
\end{aligned}
$$

The initial condition in (8.7) can be approximated by

$$
U_m^0 = \begin{cases}
\dfrac{1}{\Delta r}\left[1-\dfrac{r^*}{\Delta r}+\operatorname{int}\left(\dfrac{r^*}{\Delta r}\right)\right], & m=\operatorname{int}\left(\dfrac{r^*}{\Delta r}\right), \\
\dfrac{1}{\Delta r}\left[\dfrac{r^*}{\Delta r}-\operatorname{int}\left(\dfrac{r^*}{\Delta r}\right)\right], & m=\operatorname{int}\left(\dfrac{r^*}{\Delta r}\right)+1, \\
0, & \text{otherwise},
\end{cases} \tag{8.13}
$$

where $\operatorname{int}(x)$ is the integer part of the number x, and we assume $r^* \in [r_l + \Delta r, r_u - \Delta r]$.

By the trapezoidal rule (see Subsection 5.1.3), the condition (8.8) can be approximated by

$$
\begin{aligned}
& \lambda^{n+1/2}\Delta r\left[\frac{1}{4}w_0^{n+\frac{1}{2}}\left(U_0^{n+1}+U_0^n\right)+\tfrac{1}{2}\sum_{m=1}^{M-1}w_m^{n+\frac{1}{2}}\left(U_m^{n+1}+U_m^n\right)\right. \\
& \qquad\qquad\left.+\frac{1}{4}w_M^{n+\frac{1}{2}}\left(U_M^{n+1}+U_M^n\right)\right] \\
& +\frac{\Delta r}{4}\left[(r_0^{n+\frac{1}{2}})^2-u_0^{n+\frac{1}{2}}\right]\left(U_0^{n+1}+U_0^n\right) \\
& +\frac{\Delta r}{2}\sum_{m=1}^{M-1}\left[(r_m^{n+\frac{1}{2}})^2-u_m^{n+\frac{1}{2}}\right]\left(U_m^{n+1}+U_m^n\right) \\
& +\frac{\Delta r}{4}\left[(r_M^{n+\frac{1}{2}})^2-u_M^{n+\frac{1}{2}}\right]\left(U_M^{n+1}+U_M^n\right)=\frac{\partial^2\overline{V}(t^{n+\frac{1}{2}})}{\partial T^{*2}}.
\end{aligned} \tag{8.14}
$$

From (8.13), we can have $\{U_m^0\}$. Therefore, we can have the following procedure for $n = 0, 1, \cdots, N-1$ successively. Suppose we already have $\{U_m^n\}$. Guessing[2] $\lambda^{n+1/2}$, we can obtain $\{U_m^{n+1}\}$ by solving the system consisting of (8.12) and the boundary conditions in (8.7) at $t = t^{n+1}$. Then, we check if (8.14) holds. If not, we need to find a new guess by solving $\lambda^{n+1/2}$ from (8.14) or by other iteration methods, and obtain new $\{U_m^{n+1}\}$ and check again; if it is, we find the value $\lambda^{n+1/2}$. When this procedure is done for $n = 0, 1, \cdots, N-1$ successively, we find the values for $\lambda^{n+1/2}$, $n = 0, 1, \cdots, N-1$. Another condition that can be used to determine $\lambda^{n+1/2}$ is (8.6). The advantage of using (8.6) is to let the value of the zero-coupon bonds be exactly equal to the data from the market. In this case, we have to design an iteration method

[2] As the first guess, we can let $\lambda^{1/2} = \lambda(0)$ and $\lambda^{n+1/2} = \lambda^{n-1/2}$ for $n \neq 0$.

to find the next iterative value of $\lambda^{n+1/2}$. It is clear that if (8.7) needs to be replaced by (8.10), the procedure above is almost the same.

For the method based on (4.47), in order to do one iteration to determine $\lambda(t)$, we need to integrate the partial differential equation $n+1$ times from t^{n+1} to t^0. For the method based on (8.7), in order to do the same thing, we need to integrate the partial differential equation only once from t^n to t^{n+1}. Therefore, we pay more attention to the method based on (8.7). The only complication is that the computation based on (8.7) involves the Dirac delta function. This requires us to use more grid points in the r-direction. In order for a function $\lambda(t)$ to be used in practice, we have to check whether or not the computed zero-coupon bond values are matched with the real market data well enough. If the formulation (4.47) is adopted, then such a condition is used directly when $\lambda(t)$ is determined. Thus, no further check is needed for this case. However, when the formulation (8.7) is used, theoretically the computed zero-coupon bond values should be consistent with the real market data if (8.6) or (8.8) holds. Because there exists numerical error, this fact will be true only if the the numerical error is controlled.

8.1.3 Numerical Results on Market Prices of Risk

Here, we give two examples on numerical results of inverse problems. Because the method of solving (8.7) is faster, we only give the results obtained by this method. As an example, we take the following spot interest rate model:

$$dr = (r^{**} - r)dt + r(0.2 - r)dX, \quad r_l = 0 \le r \le r_u = 0.2,$$

where r^{**} is a constant between r_l and r_u, and $r^{**} = 0.05345$ in these examples given here. This model satisfies conditions (4.45) and (4.46), so these partial differential equation problems we are going to solve are well-posed.

Table 8.1. Comparison between given and computed bond prices

($V_{b,g}$ denotes given bond prices and V_b stands for computed bond prices)

T^*	0.5	1	2	3	5	7	10	15	20	25
$V_{b,g}$	97.36	94.80	89.86	85.18	76.55	68.79	58.60	44.85	34.335	26.283
V_b	97.36	94.80	89.86	85.18	76.55	68.79	58.60	44.85	34.333	26.279

Example 1. Suppose today's bond prices are given by the exponential function $100\mathrm{e}^{-0.05345T^*}$. According to this function, we can use the method given in the last subsection to find the market price of risk $\lambda(t)$. In Fig. 8.1, the function $\lambda(t)$ is shown. As soon as we have the market price of risk, we can compute the bond price by solving the bond equation. In Table 8.1, we list both the numerical results and the values from the given function. From the table, we see that the difference is on the third decimal place, which means