$$\begin{cases} \left[\dfrac{\partial V_{so}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{so}}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_{so}}{\partial r} - rV_{so}\right] \\ \times \left[V_{so}(r,t) - \max\left(V_s(r,t;r_{se},t),0\right)\right] = 0, \\ \dfrac{\partial V_{so}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{so}}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_{so}}{\partial r} - rV_{so} \le 0, \\ V_{so}(r,t) - \max\left(V_s(r,t;r_{se},t),0\right) \ge 0, \\ V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T),0\right), \end{cases} \tag{8.17}$$

where $t \in [0,T]$ and $r \in [r_l, r_u]$. Because $V_s(r,t;r_{se},t)$ are not given functions, we have to solve (4.60) and (4.61) with $T = t$ from $t+N$ to t when $V_{so}(r,t)$ for time t needs to be determined. Of course, this problem can also be formulated as a free-boundary problem. The reader is asked to write down the free-boundary problem for this case as an exercise.

Table 8.6. Prices of European and American swaptions with $Q = 100$

(For each N, take the value of computed r_s given in Table 8.5 as r_{se} of the options on the swap)

	$T\backslash N$	2	3	5	10
European	0.5	0.167	0.196	0.269	0.278
	1	0.276	0.288	0.499	0.490
	2	0.492	0.548	1.083	1.021
American	0.5	0.213	0.248	0.331	0.342
	1	0.450	0.474	0.731	0.722
	2	0.678	0.753	1.338	1.273

The problems (4.62) and (8.17) can be solved by the scheme (5.37) or modified (5.31). In Table 8.6, we list some numerical results on European and American swaptions. The exercise swap rates r_{se} are 0.05335, 0.05423, 0.05506, 0.05712 for $N = 2, 3, 5, 10$, respectively. The other parameters are given in the table.

8.3 Pricing Derivatives with Multi-Factor Models

8.3.1 Determining Models from the Market Data

In Section 4.6, a three-factor interest rate model was proposed. In this section, we will discuss implicit finite-difference methods for the three-factor interest rate derivative problems and some other related problems. In order to use that model to price an interest rate derivative, we need to know how to find the payoff of the derivative and to determine those coefficients in the partial differential equation (4.82). In this subsection, we will discuss these two problems, and the next subsection is devoted to implicit finite-difference methods.

Suppose we want to price a half-year option on five-year swaps with an exercise swap rate r_{se}. Assume the day we want to price the swaption (the option on swaps) to be denoted as $t = 0$. Thus, according to the notation given in Subsection 4.5.2, $T = 0.5$ and $N = 5$.

First, let us discuss how to determine the final value. On the market, the prices of 3-month, 6-month, 1-year, 2-year, 3-year and 5-year zero-coupon bonds are given every day. Set $T_1^* = 0.25, T_2^* = 0.5, T_3^* = 1, T_4^* = 2, T_5^* = 3$, and $T_6^* = 5$, let Z_i denote the price of the bond with maturity T_i^*, and define $S_i = Z_i/T_i^*$, $i = 1, 2, \cdots, 6$. Suppose we have these values on a period of L days and let $S_{i,l}$ stand for the value of S_i at the l-th day, $l = 1, 2, \cdots, L$. By b_i^2 and $b_i b_j \rho_{i,j}$, we denote the variance of S_i and the covariance between S_i and S_j, respectively. From statistics, we know that b_i^2 and $\rho_{i,j}$ can be estimated by

$$\begin{aligned} b_i^2 &= \frac{1}{L-1} \sum_{l=1}^{L} \left(S_{i,l} - \frac{1}{L} \sum_{l=1}^{L} S_{i,l} \right)^2 \\ &= \frac{1}{L-1} \left[\sum_{l=1}^{L} (S_{i,l})^2 - \frac{1}{L} \left(\sum_{l=1}^{L} S_{i,l} \right)^2 \right] \end{aligned}$$

and

$$\begin{aligned} \rho_{ij} &= \frac{\sum\limits_{l=1}^{L} \left(S_{i,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{i,l} \right) \left(S_{j,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{j,l} \right)}{\sqrt{\left[\sum\limits_{l=1}^{L} \left(S_{i,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{i,l} \right)^2 \times \sum\limits_{l=1}^{L} \left(S_{j,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{j,l} \right)^2 \right]}} \\ &= \frac{\sum\limits_{l=1}^{L} (S_{i,l} S_{j,l}) - \frac{1}{L} \left(\sum\limits_{l=1}^{L} S_{i,l} \times \sum\limits_{l=1}^{L} S_{j,l} \right)}{\sqrt{\left[\sum\limits_{l=1}^{L} (S_{i,l})^2 - \frac{1}{L} \left(\sum\limits_{l=1}^{L} S_{i,l} \right)^2 \right] \left[\sum\limits_{l=1}^{L} (S_{j,l})^2 - \frac{1}{L} \left(\sum\limits_{l=1}^{L} S_{j,l} \right)^2 \right]}}. \end{aligned}$$

Using the data for the period from January 4, 1982, to February 15, 2002, we obtain

$$\mathbf{B} = \begin{bmatrix} b_1^2 & b_1 b_2 \rho_{1,2} & \cdots & b_1 b_6 \rho_{1,6} \\ b_1 b_2 \rho_{1,2} & b_2^2 & \cdots & b_2 b_6 \rho_{2,6} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 b_6 \rho_{1,6} & b_2 b_6 \rho_{2,6} & \cdots & b_6^2 \end{bmatrix}$$

$$= 10^{-3} \begin{bmatrix} 0.4644 & 0.4758 & 0.4637 & 0.4224 & 0.3776 & 0.2993 \\ 0.4758 & 0.4916 & 0.4818 & 0.4413 & 0.3956 & 0.3145 \\ 0.4637 & 0.4818 & 0.4760 & 0.4392 & 0.3952 & 0.3161 \\ 0.4224 & 0.4413 & 0.4392 & 0.4109 & 0.3724 & 0.3014 \\ 0.3776 & 0.3956 & 0.3952 & 0.3724 & 0.3392 & 0.2766 \\ 0.2993 & 0.3145 & 0.3161 & 0.3014 & 0.2766 & 0.2289 \end{bmatrix}.$$

By the **QR** method given in Subsection 5.2.4 or other methods, we can find the eigenvalues and the unit eigenvectors of $\mathbf{B}$. As soon as we have them, $\mathbf{B}$ can be rewritten as

$$\mathbf{B} = \mathbf{A}^T \mathbf{C} \mathbf{A},$$

where

$$\mathbf{A} = \begin{bmatrix} 0.4366 & 0.4533 & 0.4479 & 0.4151 & 0.3745 & 0.3011 \\ -0.5426 & -0.3546 & -0.0918 & 0.2650 & 0.4190 & 0.5706 \\ -0.5871 & 0.1231 & 0.5461 & 0.2779 & -0.0121 & -0.5143 \\ -0.3980 & 0.6808 & 0.0016 & -0.4305 & -0.1994 & 0.3912 \\ 0.1082 & -0.4337 & 0.7019 & -0.4366 & -0.1869 & 0.2864 \\ -0.0031 & 0.0448 & 0.0113 & -0.5516 & 0.7806 & -0.2902 \end{bmatrix}$$

and

$$\mathbf{C} = 10^{-3} \times \ \text{diag}\,(2.366, 0.04109, 0.003240, \\ 3.953 \times 10^{-4}, 1.996 \times 10^{-4}, 4.498 \times 10^{-5}).$$

Because the last three components of $\mathbf{C}$ are very small compared with the first three components, the six random variables, S_1, S_2, $\cdots$, S_6, almost depend on only three variables. Because

$$\begin{vmatrix} a_{1,1} & a_{1,4} & a_{1,6} \\ a_{2,1} & a_{2,4} & a_{2,6} \\ a_{3,1} & a_{3,4} & a_{3,6} \end{vmatrix} = \begin{vmatrix} 0.4366 & 0.4151 & 0.3011 \\ -0.5426 & 0.2650 & 0.5706 \\ -0.5871 & 0.2779 & -0.5143 \end{vmatrix} \approx -0.3822 \neq 0,$$

we can choose S_1, S_4, and S_6 as the three independent components, which will be denoted by S_{i_1}, S_{i_2}, and S_{i_3} in what follows. From Subsection 4.6.2, we know that the values of S_i, $i \neq i_1, i_2$, and i_3, are uniquely determined by (4.67) for a given set of S_{i_1}, S_{i_2}, and S_{i_3} when $\mathbf{A}$ is found and $S_i^*, i = 1, 2, \cdots, 6$, are specified. [4] Based on the six values of S_1, S_2, $\cdots$, S_6, a zero-coupon bond

[4]In this way, for any day in the period from January 4, 1982, to February 15, 2002, we can obtain the theoretical values of S_2, S_3, and S_5 by giving the market data of S_1, S_4, and S_6. That is, from the market prices of 3-month, 2-year, and 5-year zero-coupon bonds we can obtain the theoretical prices of 6-month, 1-year, and 3-year zero-coupon bonds for any day. In Fig. 8.7 we compare the theoretical prices of 6-month, 1-year, and 3-year zero-coupon bonds with their market data for any day in the period from January 4, 1982, to February 15, 2002. The figure shows that the theoretical prices and the market data are very close to each other.

curve with a maximum maturity $T^*_{\max} = 5$ can be found by using the cubic spline interpolation. Assume that for the period $t \in [0, T] = [0, 0.5]$, S^*_i are constants, for example, are equal to the values of zero-coupon bonds at $t = 0$. Thus, the possible zero-coupon bond curves for any $t \in [0, T]$ are the same, i.e.,

$$\bar{Z}\left(T^*; Z_{i_1}, Z_{i_2}, Z_{i_3}, t\right) = \bar{Z}\left(T^*; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0\right).$$

Here in order to indicate the dependence of the zero-coupon bond curves on $Z_{i_1}, Z_{i_2}, Z_{i_3}$, instead of $\bar{Z}\left(T^*; t\right)$, we use $\bar{Z}\left(T^*; Z_{i_1}, Z_{i_2}, Z_{i_3}, t\right)$. As soon as we have a zero-coupon bond curve, using (4.54) with $r_s = r_{se}$:

$$Q\left[1 - \frac{r_{se}}{2}\sum_{k=1}^{2N} Z\left(T; T + k/2\right) - Z\left(T; T + N\right)\right],$$

we can determine the value of a swap with an exercise rate r_{se}. Here, Q is the notional principal and $Z(T; T + k/2) = \bar{Z}(k/2; Z_{i_1}, Z_{i_2}, Z_{i_3}, T) = \bar{Z}(k/2; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0)$. Therefore, the final value of a swaption is

$$Q \max\left(1 - \frac{r_{se}}{2}\sum_{k=1}^{2N} \bar{Z}\left(k/2; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0\right) - \bar{Z}\left(N; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0\right), 0\right). \tag{8.18}$$

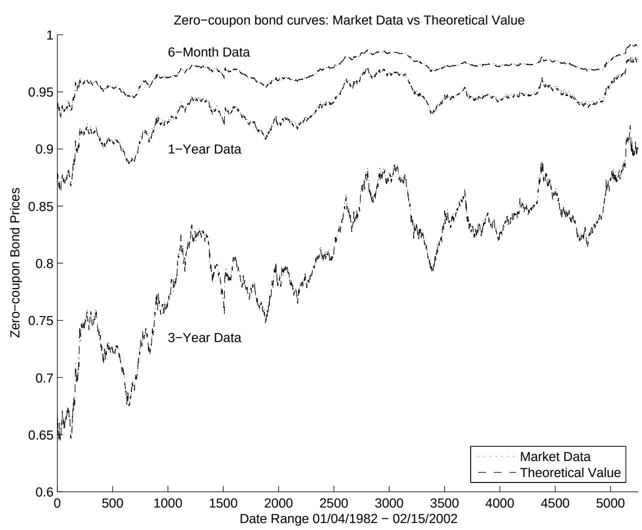

Fig. 8.7. Comparison between the market data and the theoretical values of zero-coupon bonds

Before discussing how to determine the coefficients in the partial differential equation, we would like to give some information about how these zero-coupon bond curves generated above are close to the real zero-coupon bond curves. Suppose that one day, the prices of zero-coupon bonds are

$$\begin{cases} Z_1 = 0.9811,\ Z_2 = 0.9559,\ Z_3 = 0.9047, \\ Z_4 = 0.7979,\ Z_5 = 0.7068 \quad \text{and} \quad Z_6 = 0.5475, \end{cases} \tag{8.19}$$

which correspond to the following interest rates:

$$\begin{cases} r_1 = 0.0776,\ r_2 = 0.0923,\ r_3 = 0.1027, \\ r_4 = 0.1161,\ r_5 = 0.1191 \quad \text{and} \quad r_6 = 0.1242. \end{cases}$$

Here, r_i is associated with Z_i by the following expression:

$$Z_i = (1 + r_i/2)^{-2N_i},$$

where N_i is the maturity of the i-th zero-coupon bond. From this set of data, we can determine a class of zero-coupon bond curves with $Z_{i_1}, Z_{i_2}, Z_{i_3}$ as parameters. For any day in the period from January 4, 1982, to February 15, 2002, we take the values of $Z_{i_1}, Z_{i_2}, Z_{i_3}$ as input and find a zero-coupon bond curve from the class. From the zero-coupon bond curve, we obtain the values of Z_i, $i \neq i_1, i_2$, and i_3, and the differences between the values determined from the curve and the values from the original market data. We do this for every day. The average value of the differences divided by $(1 - Z_i)$, $i \neq i_1, i_2$, and i_3, is 0.005. The same thing to the swap rate and to the value of the swaption on a 5-year swap with $r_{se} = 0.1225$ is also done. The maximum difference between the swap rates from the market curve and the model curve is 0.0004 (4 basis points), and the average difference is 0.00008 (0.8 basis points). The average error of the swaption value is 0.02 if the notional principal is 100. Therefore, we may conclude that these zero-coupon bond curves reflect the market situation.

Now let us discuss how to determine the coefficients in the partial differential equation. Suppose that derivative securities depend on $Z_{i_1}, Z_{i_2}, Z_{i_3}$, and t. Let

$$\begin{cases} \xi_1 = \dfrac{Z_{i_1} - Z_{i_1,l}}{1 - Z_{i_1,l}}, \\[2ex] \xi_2 = \dfrac{Z_{i_2} - Z_{i_2,l}}{Z_{i_1} - Z_{i_2,l}}, \\[2ex] \xi_3 = \dfrac{Z_{i_3} - Z_{i_3,l}}{Z_{i_2} - Z_{i_3,l}}, \end{cases} \tag{8.20}$$

where $Z_{i_1,l}, Z_{i_2,l}$, and $Z_{i_3,l}$ are minimums of $Z_{i_1}, Z_{i_2}, Z_{i_3}$ and we set $Z_{i_1,l} = 0.9597$, $Z_{i_2,l} = 0.7209$, and $Z_{i_3,l} = 0.4332$, which are a little less than the observed minimums 0.9634, 0.7463, and 0.4847, respectively. From Subsection 4.6.3, we know that the value of a derivative security, $V(\xi_1, \xi_2, \xi_3, t)$, satisfies (4.82), where coefficients depends on $r, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}, \tilde{\rho}_{2,3}$ besides

ξ_1, ξ_2, and ξ_3. Therefore, in order to use that equation, we have to know $r, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$. It is clear that r can be determined by the slope of zero-coupon bond curves at the left end, i.e.,

$$r\left(\xi_1, \xi_2, \xi_3, t\right) = -\frac{\partial \bar{Z}}{\partial T^*}\left(0; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0\right), \tag{8.21}$$

where

$$\begin{cases} Z_{i_1} = Z_{i_1,l} + \xi_1\left(1 - Z_{i_1,l}\right), \\ Z_{i_2} = Z_{i_2,l} + \xi_2\left[Z_{i_1,l} + \xi_1\left(1 - Z_{i_1,l}\right) - Z_{i_2,l}\right], \\ Z_{i_3} = Z_{i_3,l} + \xi_3\left\{Z_{i_2,l} + \xi_2\left[Z_{i_1,l} + \xi_1\left(1 - Z_{i_1,l}\right) - Z_{i_2,l}\right] - Z_{i_3,l}\right\}. \end{cases} \tag{8.22}$$

As we know, for $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ we need to require (4.84):

$$\begin{cases} \tilde{\sigma}_1\left(0, \xi_2, \xi_3, t\right) = \tilde{\sigma}_1\left(1, \xi_2, \xi_3, t\right) = 0, \\ \tilde{\sigma}_2\left(\xi_1, 0, \xi_3, t\right) = \tilde{\sigma}_2\left(\xi_1, 1, \xi_3, t\right) = 0, \\ \tilde{\sigma}_3\left(\xi_1, \xi_2, 0, t\right) = \tilde{\sigma}_3\left(\xi_1, \xi_2, 1, t\right) = 0. \end{cases}$$

Let us assume $\tilde{\sigma}_i$ to be in the form

$$\tilde{\sigma}_i\left(\xi_1, \xi_2, \xi_3, t\right) = \tilde{\sigma}_i\left(\xi_i\right) = \tilde{\sigma}_{i,0}\frac{1 - \left(1 - 2\xi_i\right)^2}{1 - p_i\left(1 - 2\xi_i\right)^2}, \quad i = 1, 2, 3, \tag{8.23}$$

where $\tilde{\sigma}_{i,0}$ and p_i are positive constants, and $p_i \in (0, 1)$. It is clear that in this case, condition (4.84) is fulfilled. On each day, we have the values of $Z_{i_1}, Z_{i_2}, Z_{i_3}$. Because ξ_1, ξ_2, ξ_3 are defined by (8.20), we can also have the values of ξ_1, ξ_2, ξ_3 every day. Therefore, we can find $\tilde{\sigma}_i\left(\xi_i\right)$ from the data on the market using the method described in Subsection 5.6.2 with

$$g\left(\xi_i\right) = \frac{1 - \left(1 - 2\xi_i\right)^2}{1 - p_i\left(1 - 2\xi_i\right)^2} \quad \text{and} \quad N = 0.$$

For $\tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$, there is no requirement. We assume that they are constant and that the value can also be obtained using the method described in Subsection 5.6.2.

Taking $p_1 = p_2 = p_3 = 0.8$ and using the data on the market for the period between January 4, 1982, and February 15, 2002, we obtain

$$\tilde{\sigma}_{1,0} = 0.09733, \quad \tilde{\sigma}_{2,0} = 0.08622, \quad \tilde{\sigma}_{3,0} = 0.08148$$

and

$$\tilde{\rho}_{1,2} = 0.5682, \quad \tilde{\rho}_{1,3} = 0.4996, \quad \tilde{\rho}_{2,3} = 0.8585.$$