and 6.3, the results of call and put options on several meshes are given. The method used is PEFDII. The error and the CPU time needed are also shown. In order to have an error, we must have the exact solutions. The exact solution for the American call and put option problems with these parameters are $C = 9.94092345\cdots$ and $P = 5.92827717\cdots$, which are obtained by the SSM given in Chapter 7. Here, the first nine digits are given, and it is enough to determine the first few digits of the errors given in these tables. Computation is done on a Space Ultra 10 computer. In this book, when a CPU time is mentioned, the computation is done on such a computer if no other explanation is given.

6.1.3 Binomial and Trinomial Methods

This subsection is devoted to the binomial and trinomial methods. In these methods, there is a lattice of possible asset prices. Thus, such methods are also called lattice methods.

Binomial methods. The binomial method is a simple and very effective method for computing the option prices.

When the Black-Scholes equation is derived, a risk-free portfolio is established. This idea can also be used to design numerical methods. Let S_n be the given stock price at time t^n, S_{n+1} be the stock price at time $t^{n+1} = t^n + \Delta t$, and the possible values of S_{n+1} be $S_{n+1,0}$ and $S_{n+1,1}$. Assume that the stock pays dividends continuously and the dividend yield is D_0. Therefore one share of stock at time t^n becomes $\mathrm{e}^{D_0 \Delta t}$ shares at time t^{n+1}. Let V_n be the price of a derivative at time t^n, and $V_{n+1,i}$ be the price of the derivative at time t^{n+1} if the stock price is $S_{n+1,i}$, $i = 0$ and 1. That the portfolio

$$V - \Delta S$$

is risk-free means that

$$V_{n+1,0} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,0} = V_{n+1,1} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,1} = (V_n - \Delta S_n)\, \mathrm{e}^{r \Delta t}.$$

Therefore

$$\Delta = \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \mathrm{e}^{-D_0 \Delta t}$$

and

$$\begin{aligned}
V_n &= \mathrm{e}^{-r\Delta t} \left(V_{n+1,0} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,0}\right) + \Delta S_n \\
&= \mathrm{e}^{-r\Delta t} \left(V_{n+1,0} - \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,0}\right) + \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \mathrm{e}^{-D_0 \Delta t} S_n \\
&= \mathrm{e}^{-r\Delta t} \left[\frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} V_{n+1,1} \right. \\
&\qquad \left. + \left(1 - \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}}\right) V_{n+1,0} \right].
\end{aligned}$$

Let

$$p = \frac{S_n e^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}}, \tag{6.12}$$

then we have

$$V_n = e^{-r\Delta t} \left[pV_{n+1,1} + (1-p)\, V_{n+1,0} \right]. \tag{6.13}$$

Suppose that in the real world, the stock price satisfies

$$dS = \mu S dt + \sigma S dX = \mu S dt + \sigma S \phi \sqrt{dt},$$

or

$$S_{n+1} - S_n = \mu S_n \Delta t + \sigma S_n \phi \sqrt{\Delta t},$$

where ϕ is the standardized normal random variable. Using Itô's lemma, this model can be rewtitten as

$$d \ln S = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dX = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \phi \sqrt{dt},$$

or

$$\ln S_{n+1} - \ln S_n = \left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \phi \sqrt{\Delta t}. \tag{6.14}$$

According to this model, the number of possible prices of the stock at time t_{n+1} is infinity. In the derivation above, we think that there are only two possible values the price of the stock can take at time t_{n+1}. Thus the random variable ϕ is approximated by a binomial random variable. Let ψ denote this binomial random variable. Because $\mathrm{E}[\phi] = 0$ and $\mathrm{E}[\phi^2] = \mathrm{Var}[\phi] + \mathrm{E}^2[\phi] = 1$, it is natural to require $\mathrm{E}[\psi] = 0$ and $\mathrm{E}[\psi^2] = 1$. Suppose that the two values of ψ are ψ_0 and ψ_1 and that the probabilities of taking ψ_0 and ψ_1 are $1-q$ and q, respectively. Then the two conditions can be written as

$$\begin{cases} (1-q)\,\psi_0 + q\psi_1 = 0, \\ (1-q)\,\psi_0^2 + q\psi_1^2 = 1. \end{cases}$$

From these two equations we can have

$$\begin{cases} q = \dfrac{-\psi_0}{\psi_1 - \psi_0}, \\[2ex] q = \dfrac{1-\psi_0^2}{\psi_1^2 - \psi_0^2}. \end{cases}$$

Hence

$$-\psi_0 = \frac{1-\psi_0^2}{\psi_1 + \psi_0}$$

or

$$\psi_0 \psi_1 = -1.$$

Therefore $\psi_0\psi_1 = -1$ is a necessary condition for $\mathrm{E}[\psi^2] = 1$ and $\mathrm{E}[\psi] = 0$. From the procedure of deriving this condition, it is easy to see that this condition is also a sufficient condition for $\mathrm{E}[\psi^2] = 1$ if $\mathrm{E}[\psi] = 0$. It is clear, if we choose ψ_0 and ψ_1 so that

$$\psi_0\psi_1 = -1 + O(\Delta t)$$

and require $\mathrm{E}[\psi] = 0$, then ψ is still a good approximate to ϕ.

Suppose that ψ_i is related to $S_{n+1,i}, i = 0, 1$. Thus we have

$$\begin{cases} \ln S_{n+1,0} = \ln S_n + \left(\mu - \dfrac{\sigma^2}{2}\right)\Delta t + \sigma\psi_0\sqrt{\Delta t}, \\ \ln S_{n+1,1} = \ln S_n + \left(\mu - \dfrac{\sigma^2}{2}\right)\Delta t + \sigma\psi_1\sqrt{\Delta t}. \end{cases}$$

Let us choose

$$\begin{cases} \psi_0 = -1 - \left(\mu - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{6.15}$$

Because $\psi_0\psi_1 = -1 + \left(\mu - \dfrac{\sigma^2}{2}\right)^2 \Delta t/\sigma^2$, ψ is an approximate to ϕ. In this case

$$\begin{cases} \ln S_{n+1,0} = \ln S_n - \sigma\sqrt{\Delta t}, \\ \ln S_{n+1,1} = \ln S_n + \sigma\sqrt{\Delta t}, \end{cases}$$

or

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{-\sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{\sigma\sqrt{\Delta t}}. \end{cases} \tag{6.16}$$

Using (6.12), (6.13) and (6.16), we can evaluate the price of a derivative if the stock price satisfies (6.14). This is called the binomial method which was proposed by Cox, Ross, and Rubinstein in 1979 [21].

For ψ_0 and ψ_1, we can choose other expressions. For example (see the book by McDonald [57]), let

$$\begin{cases} \psi_0 = -1 - \left(\mu - r + D_0 - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - r + D_0 - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{6.17}$$

Because $\psi_0\psi_1 = -1 + \left(\mu - r - D_0 - \dfrac{\sigma^2}{2}\right)^2 \Delta t/\sigma^2$, ψ is an approximate to ϕ. In this case

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{(r-D_0)\Delta t - \sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{(r-D_0)\Delta t + \sigma\sqrt{\Delta t}}. \end{cases} \tag{6.18}$$

Generally, we can choose

$$\begin{cases} \psi_0 = -1 - \left(\mu - c - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - c - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{6.19}$$

In this case

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{c\Delta t - \sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{c\Delta t + \sigma\sqrt{\Delta t}}, \end{cases} \tag{6.20}$$

and both (6.16) and (6.18) are in this form.

If p is determined by (6.12), then we have

$$\begin{aligned} & pS_{n+1,1} + (1-p)\, S_{n+1,0} \\ = & \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,1} + \frac{S_{n+1,1} - S_n \mathrm{e}^{(r-D_0)\Delta t}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,0} \\ = & \mathrm{e}^{(r-D_0)\Delta t} S_n. \end{aligned}$$

When $0 \le p \le 1$, this relation can be interpreted as follows. When a derivative is priced, the probability of the price at t^{n+1} being $S_{n+1,1}$ is p and the probability of the price at t^{n+1} being $S_{n+1,0}$ is $1-p$, and the expectation of the stock price at t^{n+1} is $\mathrm{e}^{(r-D_0)\Delta t} S_n$:

$$\mathrm{E}_D\left[S_{n+1}\right] = pS_{n+1,1} + (1-p)\, S_{n+1,0} = \mathrm{e}^{(r-D_0)\Delta t} S_n = \mathrm{e}^{r\Delta t}\mathrm{e}^{-D_0\Delta t} S_n, \tag{6.21}$$

where we use E_D as the notation for expectation in the case a derivative is priced. In the front of S_n there is a factor $\mathrm{e}^{-D_0\Delta t}$ because the expectation of the stock price should go down by a factor of $\mathrm{e}^{-D_0\Delta t}$ as one share of stock at time t^n becomes $\mathrm{e}^{D_0\Delta t}$ shares of stock at time t^{n+1}, and there is another factor $\mathrm{e}^{r\Delta t}$ because the expectation of the stock price should go up by a factor of $\mathrm{e}^{r\Delta t}$ just like any risk-free investment. Because of this, we usually say that $\mathrm{E}_D\left[S_{n+1}\right]$ is the expectation of S_{n+1} in the "risk-neutral" world. According to the model for the stock price, we have

$$\mathrm{E}[S_{n+1}] = S_n + \mu S_n \Delta t = \left(\mathrm{e}^{\mu\Delta t} + O(\Delta t^2)\right) S_n.$$

That is, in the expression for the expectation of the stock price at time t_{n+1} in the real world, there is a factor about $\mathrm{e}^{\mu\Delta t}$ in the front of S_n, which is completely different from the case when we price derivatives.

When $S_{n+1,0}$ and $S_{n+1,1}$ are given by (6.16), then

$$p=\frac{S_n \mathrm{e}^{(r-D_0)\Delta t}-S_n \mathrm{e}^{-\sigma\sqrt{\Delta t}}}{S_n \mathrm{e}^{\sigma\sqrt{\Delta t}}-S_n \mathrm{e}^{-\sigma\sqrt{\Delta t}}}=\frac{\mathrm{e}^{(r-D_0)\Delta t}-\mathrm{e}^{-\sigma\sqrt{\Delta t}}}{\mathrm{e}^{\sigma\sqrt{\Delta t}}-\mathrm{e}^{-\sigma\sqrt{\Delta t}}} \tag{6.22}$$

and $0 \le p \le 1$ is equivalent to $\mathrm{e}^{-\sigma\sqrt{\Delta t}} \le \mathrm{e}^{(r-D_0)\Delta t} \le \mathrm{e}^{\sigma\sqrt{\Delta t}}$. The inequality $\mathrm{e}^{(r-D_0)\Delta t} \le \mathrm{e}^{\sigma\sqrt{\Delta t}}$ might not hold for large Δt and p does not represent a probability in this case. However this case usually does not occur in practice because Δt would be small in real computation. When $S_{n+1,0}$ and $S_{n+1,1}$ are given by (6.18), then

$$p=\frac{S_n \mathrm{e}^{(r-D_0)\Delta t}-S_n \mathrm{e}^{(r-D_0)\Delta t-\sigma\sqrt{\Delta t}}}{S_n \mathrm{e}^{(r-D_0)\Delta t+\sigma\sqrt{\Delta t}}-S_n \mathrm{e}^{(r-D_0)\Delta t-\sigma\sqrt{\Delta t}}}=\frac{1-\mathrm{e}^{-\sigma\sqrt{\Delta t}}}{\mathrm{e}^{\sigma\sqrt{\Delta t}}-\mathrm{e}^{-\sigma\sqrt{\Delta t}}} \tag{6.23}$$

and $0 \le p \le 1$ always holds. Hence in this case p can always be interpreted as the probability of the price being $S_{n+1,1}$ at t_{n+1}.

In the "risk-neutral" world, the variance of S_{n+1} is

$$\begin{aligned}
&\mathrm{Var}_D\left[S_{n+1}\right]\\
&=\frac{S_n \mathrm{e}^{(r-D_0)\Delta t}-S_{n+1,0}}{S_{n+1,1}-S_{n+1,0}}\left(S_{n+1,1}-\mathrm{e}^{(r-D_0)\Delta t}S_n\right)^2\\
&\quad+\frac{S_{n+1,1}-S_n \mathrm{e}^{(r-D_0)\Delta t}}{S_{n+1,1}-S_{n+1,0}}\left(S_{n+1,0}-\mathrm{e}^{(r-D_0)\Delta t}S_n\right)^2\\
&=\left(S_n \mathrm{e}^{(r-D_0)\Delta t}-S_{n+1,0}\right)\left(S_{n+1,1}-S_n \mathrm{e}^{(r-D_0)\Delta t}\right)\\
&=S_n^2 \mathrm{e}^{2(r-D_0)\Delta t}\cdot\left(1-\frac{S_{n+1,0}}{S_n \mathrm{e}^{(r-D_0)\Delta t}}\right)\left(\frac{S_{n+1,1}}{S_n \mathrm{e}^{(r-D_0)\Delta t}}-1\right)\\
&=S_n^2 \mathrm{e}^{2(r-D_0)\Delta t}\cdot\left(\frac{S_{n+1,0}}{S_n \mathrm{e}^{(r-D_0)\Delta t}}+\frac{S_{n+1,1}}{S_n \mathrm{e}^{(r-D_0)\Delta t}}-\frac{S_{n+1,0}S_{n+1,1}}{S_n^2 \mathrm{e}^{2(r-D_0)\Delta t}}-1\right).
\end{aligned}$$

When $S_{n+1,0}$ and $S_{n+1,1}$ are given by (6.20), both (6.16) and (6.18) being in this form, the expression above can further be written as:

$$\begin{aligned}
&\mathrm{Var}_D\left[S_{n+1}\right]\\
&=S_n^2 \mathrm{e}^{2(r-D_0)\Delta t}\left(\mathrm{e}^{-(r-D_0-c)\Delta t-\sigma\sqrt{\Delta t}}+\mathrm{e}^{-(r-D_0-c)\Delta t+\sigma\sqrt{\Delta t}}\right.\\
&\qquad\qquad\left.-\mathrm{e}^{-2(r-D_0-c)\Delta t}-1\right)\\
&=S_n^2 \mathrm{e}^{(r-D_0+c)\Delta t}\left(\mathrm{e}^{-\sigma\sqrt{\Delta t}}+\mathrm{e}^{\sigma\sqrt{\Delta t}}-\mathrm{e}^{-(r-D_0-c)\Delta t}-\mathrm{e}^{(r-D_0-c)\Delta t}\right)\\
&=S_n^2 \mathrm{e}^{(r-D_0+c)\Delta t}\left[1-\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t-\frac{1}{6}\sigma^3\Delta t^{3/2}+1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t\right.\\
&\quad\left.+\frac{1}{6}\sigma^3\Delta t^{3/2}-1+(r-D_0-c)\Delta t-1-(r-D_0-c)\Delta t+O\left(\Delta t^2\right)\right]\\
&=S_n^2 \mathrm{e}^{(r-D_0+c)\Delta t}\left[\sigma^2\Delta t+O\left(\Delta t^2\right)\right]\\
&=S_n^2\sigma^2\Delta t+O\left(\Delta t^2\right).
\end{aligned} \tag{6.24}$$

In the real world,

$$\mathrm{Var}[S_{n+1}] = \mathrm{Var}\left[S_n + \mu S_n \Delta t + \sigma S_n \phi \sqrt{\Delta t}\right] = \sigma^2 S_n^2 \Delta t.$$

Therefore as $\Delta t \to 0$ the variance of S_{n+1} in the "risk-neutral" world will tend to the variance of S_{n+1} in the real world.

Now let us describe the complete method proposed by Cox, Ross, and Rubinstein [21]. Define

$$d = \mathrm{e}^{-\sigma\sqrt{\Delta t}} \tag{6.25}$$

and

$$u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\Delta t}}, \tag{6.26}$$

then $S_{n+1,1} = S_n u$, $S_{n+1,0} = S_n d$, and (6.22) and (6.13) can be rewritten as .

$$p = \frac{\mathrm{e}^{(r-D_0)\Delta t} - d}{u - d}. \tag{6.27}$$

and

$$\begin{aligned} &V(S_n, n\Delta t) \\ &= \mathrm{e}^{-r\Delta t}\left[pV\left(S_{n+1,1}, (n+1)\Delta t\right) + (1-p)V\left(S_{n+1,0}, (n+1)\Delta t\right)\right]. \end{aligned} \tag{6.28}$$

Here $V(S,t)$ is the value of an option.

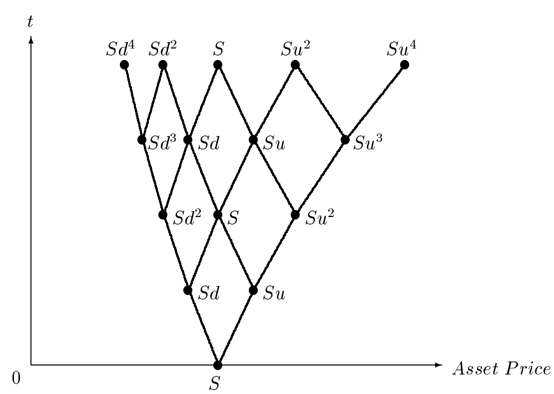

Fig. 6.2. Tree of asset prices for a binomial model

Suppose the asset price at the current time t to be S, and we divide the remaining life of the derivative security into N equal time subintervals with time step $\Delta t = (T-t)/N$. At the first time level $t + \Delta t$, there are two possible asset prices Su and $Sd = Su^{-1}$. At the second time level $t + 2\Delta t$, there are three possible asset prices, Su^2, $Sud = Sdu = S$, and $Sd^2 = Su^{-2}$. At the third time level $t+3\Delta t$, there are four possible asset prices, Su^3, $Su^2d = Su$, $Sud^2 = Su^{-1}$, and $Sd^3 = Su^{-3}$. In general, at the n-th time level $t+n\Delta t$, there are $n+1$ possible values of the asset price. Originally, at the n-th time level, there should be 2^n possible values of the asset price. However since $d = 1/u$ is required, many points are the same. For example, S, Su^2d^2, Su^4d^4, $\cdots$ are the same point because $d = 1/u$. Hence the number of possible values is greatly reduced. Let $S_{n,m}, m = 0, 1, \cdots, n$, denote the $n + 1$ possible values of the asset price at the n-th time level from the smallest to the largest. Then

$$S_{n,m} = Su^{2m-n}, \ m = 0, 1, \cdots, n. \tag{6.29}$$

For $N = 4$, all the possible prices for each n are given in Fig. 6.2. This plot is usually referred to as a tree or lattice of possible asset prices.

Assuming that we know the payoff function for our derivative security and that it depends only on the values of the underlying asset at expiry, this enables us to value it at expiry, the N-th time level. If we are considering a call, for example, we find

$$c_{N,m} = \max(S_{N,m} - E, 0), \quad m = 0, 1, \cdots, N, \tag{6.30}$$

where E is the exercise price and $c_{N,m}$ denotes the value of the call for the m-th possible asset value $S_{N,m}$ at time-step N. For a put, we know that

$$p_{N,m} = \max(E - S_{N,m}, 0), \quad m = 0, 1, \cdots, N, \tag{6.31}$$

where $p_{N,m}$ denotes the value of the put for the m-th possible asset value $S_{N,m}$ at expiry.

We can now find the expected value of the derivative security at the $(N-1)$-th time level and for possible asset prices $S_{N-1,m}$, $m = 0, 1, ..., N-1$ because we know that the probability of an asset price moving from $S_{N-1,m}$ to $S_{N,m+1}$ during a time step is p and that the probability of it moving to $S_{N,m}$ is $(1-p)$. Using the discounting factor $e^{-r\Delta t}$, we can obtain the value of the security at each possible asset price for the $(N-1)$-th time level. This procedure can be applied to the n-th time level if the values of the option for the $(n+1)$-th time level have been obtained, and the computational formula is (6.28) or, in a general form,

$$V_{n,m} = e^{-r\Delta t}(pV_{n+1,m+1} + (1-p)V_{n+1,m}), \quad m = 0, 1, \cdots, n. \tag{6.32}$$

Here, $V_{n,m}$ denotes the value of a European option at the n-th time level and corresponding to asset price $S_{n,m}$. According to this formula, starting from the payoff function, $V_{N,m}$, $m = 0, 1, \cdots, N$, we can recursively determine $V_{n,m}, m = 0, 1, \cdots, n$ for $n = N-1, N-2, \cdots, 0$, and the final value $V_{0,0}$ is the current value of the option.

For American options, we can easily incorporate the possibility of early exercise of an option into the binomial model. Because the price of an American call option must be greater than or equal to

$$\max(S_{n,m} - E, 0), \tag{6.33}$$

when calculating the price of an American call option, we need to replace (6.32) by

$$C_{n,m} = \max\left(S_{n,m} - E,\ 0,\ \mathrm{e}^{-r\Delta t}\left[pC_{n+1,m+1} + (1-p)C_{n+1,m}\right]\right) \tag{6.34}$$

at each point. Similarly, for an American put option, the formula is

$$P_{n,m} = \max\left(E - S_{n,m},\ 0,\ \mathrm{e}^{-r\Delta t}\left[pP_{n+1,m+1} + (1-p)P_{n+1,m}\right]\right) \tag{6.35}$$

because the price of an American put option has to be at least

$$\max(E - S_{n,m}, 0). \tag{6.36}$$

From what has been described, we see that the entire computation can be done in two steps. In the first step, we calculate all the $S_{n,m}$ to be used. Then, we find $V_{N,m}, m = 0, 1, \cdots, N$ and calculate $V_{n,m}, m = 0, 1, \cdots, n$ for $n = N-1, N-2, \cdots, 0$ successively. When a European option is calculated, only the $S_{N,m}, m = 0, 1, \cdots, N$, are used in order to find $V_{N,m}$. When an American option is evaluated, all the $S_{n,m}$ are needed. However, because $S_{n,m} = Su^{2m-n} = Su^{2(m-1)-(n-2)} = S_{n-2,m-1}$, we indeed only need to calculate $S_{N,m}, m = 0, 1, \cdots, N$ and $S_{N-1,m}, m = 0, 1, \cdots, N-1$, i.e., $Su^m, m = -N, -N+1, \ldots, N$. For this method, the total number of nodes is $(N+2)(N+1)/2$, so the execution time for computing all the $V_{n,m}$ is $O(N^2)$.

If the method given in the book by McDonald [57] wants to be adopted, instead of (6.25)-(6.26), (6.18) and (6.23) should be used. Also the tree of asset prices is different. In this case we should define

$$S_{n,m} = S\mathrm{e}^{n(r-D_0)\Delta t}u^{2m-n},\ m = 0, 1, \cdots, n$$

with $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$.

Trinomial methods. If σ depends on S, then u is not a constant. In this case, generally speaking, at the n-th time level, there are 2^n possible values of the asset prices that need to be considered, and the total nodes and the execution time will be very large if a binomial method is used. In order to reduce the nodes for a problem with variable σ, we can use trinomial methods. In a trinomial method, given a current asset value S, the asset value after a time-step Δt can take any of the three values

$$Su,\ Sq,\ Sd,$$

where $0 \le d < q < u$. Let p_u be the probability of the value of the asset after a time-step Δt being Su, p_q be the probability of the value being Sq, and p_d

be the probability of the value being Sd. Because there are only three possible cases, we must have

$$p_u + p_q + p_d = 1, \quad 0 \le p_u \le 1, \quad 0 \le p_q \le 1, \quad 0 \le p_u \le 1.$$

From (6.21) and (6.24), we know that in the case a derivative is priced,

$$\mathrm{E}_{D}\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$$

and

$$\begin{aligned}
\mathrm{E}_{D}\left[S_{n+1}^2\right] &= \mathrm{Var}_{D}\left[S_{n+1}\right] + \left(\mathrm{E}_{D}\left[S_{n+1}\right]\right)^2 \\
&= S_n^2 \sigma^2 \Delta t + O(\Delta t^2) + \mathrm{e}^{2(r-D_0)\Delta t} S_n^2 \\
&= \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2 + O(\Delta t^2).
\end{aligned}$$

Therefore for p_u, p_q and p_d, we can require[1]

$$\begin{aligned}
p_u u + p_q q + p_d d &= \mathrm{e}^{(r-D_0)\Delta t}, \\
p_u u^2 + p_q q^2 + p_d d^2 &= \mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t}.
\end{aligned}$$

Because there are three equations above for six unknowns, u, q, d, p_u, p_q, p_d, we can choose three parameters. In order for the number of the possible asset prices to not be 3^n at the n-th time level, we can choose

$$d = 1/u \quad \text{and} \quad q = 1. \tag{6.37}$$

Now there are only four parameters u, p_u, p_q, p_d left. They should satisfy the three conditions above. If u is given, then this is a linear system for p_u, p_q, p_d and can be solved for them easily. Its solution is

$$\begin{cases}
p_u = \dfrac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(q+d) + qd}{(u-q)(u-d)}, \\[2ex]
p_q = \dfrac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(d+u) + du}{(q-d)(q-u)}, \\[2ex]
p_d = \dfrac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(u+q) + uq}{(d-u)(d-q)}.
\end{cases} \tag{6.38}$$

Because they represent probabilities, we need to choose such a u that p_u, p_q and p_d all are nonnegative. If σ depends on S and t, then p_u, p_q and p_d will be different at different points. In this case, we need to choose such a u that at all the points p_u, p_q and p_d are nonnegative and (6.38) can still be used.

[1] We also know that because the Black–Scholes equation holds, $\mathrm{E}_{D}\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$ and $\mathrm{E}_{D}\left[S_{n+1}^2\right] = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2$ should be true (see Problem 22 of Chapter 2).

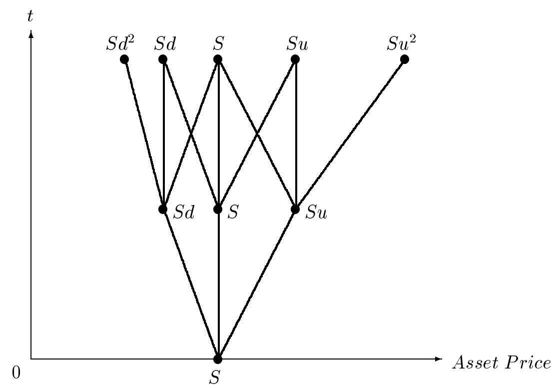

Fig. 6.3. Lattice generated by a trinomial model

The details for evaluating derivative securities using a trinomial method are nearly identical to the binomial method. The only major difference is that the expected value of the security at the n-th time level depends on the three possible values at the $(n+1)$-th time level, and that at the n-th time level, rather than $n+1$, there are $2n+1$ possible asset prices, which are

$$S_{n,m} = Su^m, m = -n, -n+1, \cdots, n.$$

In this case, the corresponding lattice is illustrated in Fig. 6.3. Let $V_{n,m}$ be the security price at $S_{n,m}$. Then, the formula for finding the expected value of a security at time level $n+1$ is

$$\mathrm{E}_D\left[V_{n+1,m}\right] = p_u V_{n+1,m+1} + p_q V_{n+1,m} + p_d V_{n+1,m-1}$$

and the value of a European derivative security for $S_{n,m}$ is

$$V_{n,m} = \mathrm{e}^{-r\Delta t}(p_u V_{n+1,m+1} + p_q V_{n+1,m} + p_d V_{n+1,m-1}),$$

and for American puts and calls we have

$$P_{n,m} = \max\left(E - S_{n,m},\ 0,\ \mathrm{e}^{-r\Delta t}\left[p_u P_{n+1,m+1} + p_q P_{n+1,m} + p_d P_{n+1,m-1}\right]\right), \tag{6.39}$$

$$C_{n,m} = \max\left(S_{n,m} - E,\ 0,\ \mathrm{e}^{-r\Delta t}\left[p_u C_{n+1,m+1} + p_q C_{n+1,m} + p_d C_{n+1,m-1}\right]\right). \tag{6.40}$$

In Tables 6.4 and 6.5, we give binomial lattice approximations to American call and put options when (6.25)–(6.28) are used. The errors and the CPU times on a computer are also shown.

Table 6.4. American call option (binomial method (6.25)–(6.28))

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1$ year, $S = E = 100$, and the exact solution is $C = 9.94092345\cdots$)

Numbers of time steps	Results	|Errors|	CPU(sec.)
50	9.902969	0.037955	0.0004
100	9.921921	0.019002	0.0013
200	9.931416	0.009507	0.0053
400	9.936168	0.004755	0.0220
800	9.938546	0.002378	0.0890

Table 6.5. American put option (binomial method (6.25)–(6.28))

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1$ year, $S = E = 100$, and the exact solution is $P = 5.92827717\cdots$)

Numbers of time steps	Results	|Errors|	CPU(sec.)
50	5.911020	0.017257	0.0004
100	5.920066	0.008211	0.0014
200	5.924273	0.004005	0.0053
400	5.926323	0.001955	0.0210
800	5.927309	0.000968	0.0880

6.1.4 Relations Between the Lattice Methods and the Explicit Finite-Difference Methods

From the view point of PDEs, the procedure given by (6.12), (6.13), and (6.20) can be understood in the following way. The value of any derivative, V, satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - rV = 0.$$

Let $\bar{S} = S\mathrm{e}^{-ct}$ and $\overline{V}\left(\bar{S}, t\right) = V(S, t)$. Since

$$\frac{\partial V}{\partial S} = \frac{\partial \overline{V}}{\partial \bar{S}} \mathrm{e}^{-ct},$$
$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 \overline{V}}{\partial \bar{S}^2} \mathrm{e}^{-2ct},$$

and

$$\frac{\partial V}{\partial t} = \frac{\partial \overline{V}}{\partial t} + \frac{\partial \overline{V}}{\partial \bar{S}} S\mathrm{e}^{-ct} \cdot (-c),$$

we have

$$\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2 \bar{S}^2 \frac{\partial^2 \overline{V}}{\partial \bar{S}^2} + (r - D_0 - c)\, \bar{S} \frac{\partial \overline{V}}{\partial \bar{S}} - r\overline{V} = 0.$$

Furthermore let us set $x = \ln \bar{S}$ and $\tilde{V}(x, t) = \overline{V}\left(\bar{S}, t\right)$. Noticing

$$\frac{\partial \overline{V}}{\partial \bar{S}} = \frac{\partial \tilde{V}}{\partial x}\frac{1}{\bar{S}},$$
$$\frac{\partial^2 \overline{V}}{\partial \bar{S}^2} = \frac{\partial^2 \tilde{V}}{\partial x^2}\frac{1}{\bar{S}^2} - \frac{1}{\bar{S}^2}\frac{\partial \tilde{V}}{\partial x},$$

and

$$\frac{\partial \overline{V}}{\partial t} = \frac{\partial \tilde{V}}{\partial t},$$

we arrived at

$$\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{V}}{\partial x^2} + \left(r - D_0 - c - \sigma^2/2\right)\frac{\partial \tilde{V}}{\partial x} - r\tilde{V} = 0. \tag{6.41}$$

For this equation, we can have the following finite difference scheme

$$\begin{aligned}&\frac{\tilde{V}_m^{n+1} - \tilde{V}_m^n}{\Delta t} + \frac{1}{2}\sigma^2 \frac{\tilde{V}_{m+1}^{n+1} - 2\tilde{V}_m^{n+1} + \tilde{V}_{m-1}^{n+1}}{\Delta x^2}\\ &+ \left(r - D_0 - c - \sigma^2/2\right)\frac{\tilde{V}_{m+1}^{n+1} - \tilde{V}_{m-1}^{n+1}}{2\Delta x} - r\tilde{V}_m^n = 0,\end{aligned}$$

or

$$\begin{aligned}\tilde{V}_m^n = \frac{1}{1 + r\Delta t}&\left[\left(\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} + \frac{r - D_0 - c - \sigma^2/2}{2}\frac{\Delta t}{\Delta x}\right)\tilde{V}_{m+1}^{n+1}\right.\\ &+ \left(1 - \frac{\sigma^2 \Delta t}{\Delta x^2}\right)\tilde{V}_m^{n+1}\\ &\left.+ \left(\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} - \frac{r - D_0 - c - \sigma^2/2}{2}\frac{\Delta t}{\Delta x}\right)\tilde{V}_{m-1}^{n+1}\right].\end{aligned} \tag{6.42}$$

Here $\tilde{V}_m^n$ denotes the value of $\tilde{V}$ at $x_m = \bar{x} + m\Delta x$ and $t^n = n\Delta t$. If we choose

$$\Delta x = \sigma\sqrt{\Delta t}, \tag{6.43}$$

then we have

$$\begin{aligned}\tilde{V}_m^n = \frac{1}{1 + r\Delta t}&\left[\left(\frac{1}{2} + \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}\right)\tilde{V}_{m+1}^{n+1}\right.\\ &\left.+ \left(\frac{1}{2} - \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}\right)\tilde{V}_{m-1}^{n+1}\right].\end{aligned} \tag{6.44}$$

Now we show that a trinomial method (a binomial method) is close to an explicit method (6.42) (an explicit method (6.44)). First we will show that the mesh here can overlop the lattices of trinomial and binomial methods. Consider the case $c = 0$ and let $\bar{x} = \ln S^*$, S^* being the asset price at the current time. In this case

$$S(x_m) = \mathrm{e}^{\bar{x}+m\Delta x} = S^* \left(\mathrm{e}^{\Delta x}\right)^m .$$

Therefore, a uniform mesh on (x,t)-plane (see Fig. 6.4) corresponds to a non-uniform mesh on (S,t)-plane (see Fig. 6.5), which overlops the lattices in Fig. 6.2 and Fig. 6.3 with $u = \mathrm{e}^{\Delta x}$ and $S = S^*$. Consequently, this explicit difference method can be understood as a trinomial method with a lattice in Fig. 6.3 and as a binomial method with a lattice in Fig. 6.2 if (6.43) holds.

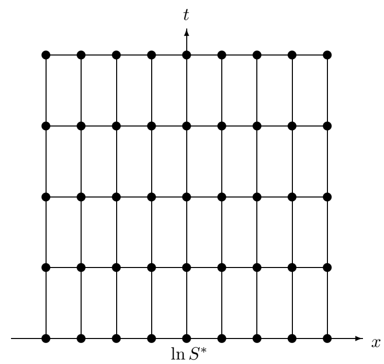

Fig. 6.4. A uniform mesh on (x,t)-plane

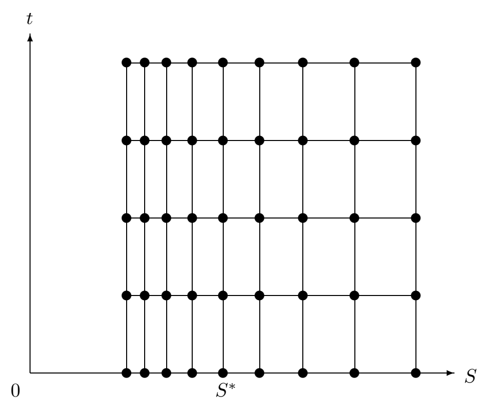

Fig. 6.5. The mesh on (S,t)-plane corresponding to a uniform mesh on (x,t)-plane

Now let show that the difference between (6.13) and (6.44) is very small. Let x_m, S_m^n, and $\bar{S}_m^n$ denote the x-coordinates, S-coordinates, and $\bar{S}$-coordinates of the m-point at time t^n, respectively. Because $x_{m+1} = x_m +$

$\Delta x = x_m+\sigma\sqrt{\Delta t}$, which means $\ln \bar{S}_{m+1}^{n+1} = \ln \bar{S}_m^n+\sigma\sqrt{\Delta t}$ or $\ln\left(S_{m+1}^{n+1}\mathrm{e}^{-ct^{n+1}}\right)$ $= \ln\left(S_m^n\mathrm{e}^{-ct^n}\right)+\sigma\sqrt{\Delta t}$, we have

$$S_{m+1}^{n+1} = S_m^n\mathrm{e}^{c\left(t^{n+1}-t^n\right)+\sigma\sqrt{\Delta t}} = S_m^n\mathrm{e}^{c\Delta t+\sigma\sqrt{\Delta t}}.$$

Similarly,

$$S_{m-1}^{n+1} = S_m^n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}.$$

Noticing that $S_{m+1}^{n+1}, S_{m-1}^{n+1}$ and S_m^n correspond to $S_{n+1,1}, S_{n+1,0}$ and S_n, we have the relations (6.20). Therefore from (6.12), we have

$$\begin{aligned}
p &= \frac{S_n\mathrm{e}^{(r-D_0)\Delta t}-S_n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}}{S_n\mathrm{e}^{c\Delta t+\sigma\sqrt{\Delta t}}-S_n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}} = \frac{\mathrm{e}^{(r-D_0-c)\Delta t}-\mathrm{e}^{-\sigma\sqrt{\Delta t}}}{\mathrm{e}^{\sigma\sqrt{\Delta t}}-\mathrm{e}^{-\sigma\sqrt{\Delta t}}}\\
&= \frac{1+(r-D_0-c)\,\Delta t-\left(1-\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t-\frac{1}{6}\sigma^3\Delta t^{3/2}\right)+O\left(\Delta t^2\right)}{2\sigma\sqrt{\Delta t}+\frac{1}{3}\sigma^3\Delta t^{3/2}+O\left(\Delta t^2\right)}\\
&= \frac{\sigma\sqrt{\Delta t}\left[1+\left(r-D_0-c-\sigma^2/2\right)\sqrt{\Delta t}/\sigma+\frac{1}{6}\sigma^2\Delta t+O\left(\Delta t^{3/2}\right)\right]}{2\sigma\sqrt{\Delta t}\left[1+\frac{1}{6}\sigma^2\Delta t+O\left(\Delta t^{3/2}\right)\right]}\\
&= \frac{1}{2}\left[1+\left(r-D_0-c-\sigma^2/2\right)\sqrt{\Delta t}/\sigma+\frac{1}{6}\sigma^2\Delta t+O\left(\Delta t^{3/2}\right)\right]\\
&\quad\times\left[1-\frac{1}{6}\sigma^2\Delta t+O\left(\Delta t^{3/2}\right)\right]\\
&= \frac{1}{2}\left[1+\frac{r-D_0-c-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right]+O\left(\Delta t^{3/2}\right).
\end{aligned}$$

Also the difference between $\mathrm{e}^{-r\Delta t}$ and $\frac{1}{1+r\Delta t}$ is $O(\Delta t^2)$. Thus (6.13) is almost the same as (6.44). Consequently, the method given by (6.12), (6.13), and (6.20) is almost an explicit scheme (6.44). Therefore, the binomial method and the trinomial method can be understood as explicit finite-difference methods in some sense.

Finally we point out that because the convergence of the explicit scheme here with $\Delta t/\Delta x^2 = \sigma^{-2}$ can be easily proved, the difference between $\frac{1}{2}\left[1+\frac{r-D_0-c-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right]$ and p is $O\left(\Delta t^{3/2}\right)$, and the difference between $\mathrm{e}^{-r\Delta t}$ and $\frac{1}{1+r\Delta t}$ is $O(\Delta t^2)$, the convergence of the binomial method can also be proved.

6.1.5 Examples of Unstable Schemes

As has been pointed out in Subsection 6.1.1, when the scheme (6.3) or (6.6) is used, stability condition (6.4) or (6.7) is required. What will happen if these conditions are violated?