MATH 6204 **Test I** **Spring 2005**

Name :____________________

ID :______________________

Show all the details of your work !!

1. (20%) Choose two problems from the following four problems:

   (a) Suppose $x_m = m\Delta x$, $y_m = m\Delta y$, and $\tau^n = n\Delta\tau$. Find the error of each of the following approximations :

   - $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{3u(x_m, \tau^n) - 4u(x_{m-1}, \tau^n) + u(x_{m-2}, \tau^n)}{2\Delta x}$;
   - $\dfrac{\partial^2 u}{\partial x \partial y}(x_m, y_l, \tau^n) \approx \dfrac{1}{2\Delta x}\left[\dfrac{u(x_{m+1}, y_{l+1}, \tau^n) - u(x_{m+1}, y_{l-1}, \tau^n)}{2\Delta y} - \dfrac{u(x_{m-1}, y_{l+1}, \tau^n) - u(x_{m-1}, y_{l-1}, \tau^n)}{2\Delta y}\right]$.

   (b) Suppose $f(x) = 0$ is a nonlinear equation. Derive the iteration formulae of Newton's method and the secant method for solving the nonlinear equation.

   (c) The heat equation

   $$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial x^2}$$

   can be approximated by the explicit first-order scheme

   $$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}$$

   or the implicit second-order scheme (the Crank–Nicolson scheme)

   $$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right).$$

   When do we choose the explicit first-order scheme and when do we use the implicit second-order scheme? Why should we choose the implicit second-order scheme if we need highly accurate results?

   (d) Suppose r and D_0 are constant and $\sigma = \sigma(S)$. Derive the symmetry relations for Bermudan options.

2. (20%) Derive the formulae of the LU decomposition method for the following almost tridiagonal system

$$\mathbf{A}\mathbf{x} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & d_1 \\ a_2 & b_2 & c_2 & & 0 & d_2 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & a_{m-1} & b_{m-1} & d_{m-1} \\ & & & & a_m & d_m \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}.$$

3. (20%) For the scheme with variable coefficients

$$\begin{aligned} &\frac{u_m^{n+1} - u_m^n}{\Delta\tau} \\ = \ & \frac{1}{4}[x_m(1-x_m)\bar{\sigma}_m]^2 \left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} \right) \\ &+ \frac{1}{2}(r - D_0)x_m(1-x_m) \left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} \right) \\ &- \frac{1}{2}\left[r\left(1-x_m\right) + D_0 x_m\right]\left(u_m^{n+1} + u_m^n\right), \end{aligned}$$

show that the condition $|\lambda_\theta(x_m, \tau^n)| \le 1 + O(\Delta\tau)$ is satisfied for any $x \in [0, 1]$.

4. (20%) As we know, an American lookback strike put option is the

solution of the following linear complementarity problem:

$$\begin{cases} \left(\dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W\right) \cdot [W(\eta, t) - \max(\eta - \beta, 0)] = 0, & 1 \le \eta, \quad t \le T, \\ \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W \le 0, & 1 \le \eta, \quad t \le T, \\ W(\eta, t) - \max(\eta - \beta, 0) \ge 0, & 1 \le \eta, \quad t \le T, \\ W(\eta, T) = \max(\eta - \beta, 0), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}(1, t) = 0, & t \le T, \end{cases}$$

where we assume that $\beta \ge 1$ and the operator $\mathbf{L}_\eta$ is defined by

$$\mathbf{L}_\eta \equiv \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + (D_0 - r)\,\eta\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem on $[0, 1]$ and with an initial condition, and design a second-order implicit method for solving the new problem. (You do not need to discuss how to solve the system.)

5. (20%) Suppose that for $V(S, t)$ there is the following jump condition:

$$V(S, t_i^-) = V(S - D_i(S), t_i^+).$$

$V_0(S, t)$ is a smooth function. Let

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ \tau = T - t, \\ \overline{V}(\xi, \tau) = \dfrac{V(S, t) - V_0(S, t)}{S + P_m}, \end{cases}$$

where P_m is a positive constant. Derive the jump condition for the function $\overline{V}(\xi, \tau)$.