MATH 6204 **Test I** **Spring 2006**

Name :______________________

ID :_________________________

Show all the details of your work !!

1. (15%) Suppose $x_m = m\Delta x$ and $\tau^n = n\Delta\tau$. Find the concrete expression of error for each of the following approximations by using the Taylor series:

   (a) $u(x_m, \tau^{n+1/2}) \approx \dfrac{u(x_m, \tau^{n+1}) + u(x_m, \tau^n)}{2}$;

   (b) $\dfrac{\partial u}{\partial \tau}(x_m, \tau^n) \approx \dfrac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}$;

   (c) $\dfrac{\partial^2 u}{\partial x^2}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - 2u(x_m, \tau^n) + u(x_{m-1}, \tau^n)}{\Delta x^2}$.

2. (20%)

   (a) Consider the implicit scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right),$$
$$m = 1, 2, \cdots, M-1$$

   with $u_0^{n+1} = f_l(\tau^{n+1})$ and $u_M^{n+1} = f_u(\tau^{n+1})$. Show that it is always stable with respect to initial values in L_2 norm. (Suppose $a > 0$.)

   (b) Show that if

$$\max_{0 \le m \le M} \frac{x_m^2(1-x_m)^2\bar{\sigma}_m^2}{2}\frac{\Delta\tau}{\Delta x^2} \le \frac{1}{2},$$

   then for the scheme with variable coefficients

$$\begin{aligned}\frac{u_m^{n+1} - u_m^n}{\Delta\tau} &= \frac{1}{2}[x_m(1-x_m)\bar{\sigma}_m]^2\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} \\ &\quad + (r - D_0)x_m(1-x_m)\frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} \\ &\quad - [r(1-x_m) + D_0 x_m]\,u_m^n,\end{aligned}$$

   the condition $|\lambda_\theta(x_m, \tau^n)| \le 1 + O(\Delta\tau)$ is satisfied for any $x_m = m/M \in [0, 1]$.

3. (15%) Derive the formulae of the LU decomposition method for the following almost tridiagonal system

$$\mathbf{A}\mathbf{x} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & d_1 \\ a_2 & b_2 & c_2 & & 0 & d_2 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & a_{m-1} & b_{m-1} & d_{m-1} \\ & & & & a_m & d_m \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}.$$

4. (15%) As we know, an American lookback strike call option is the solution of the following linear complementarity problem:

$$\begin{cases} \left(\dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W\right) \cdot \left[W(\eta, t) - \max(\alpha - \eta, 0)\right] = 0, & 0 \le \eta \le 1, \quad t \le T, \\ \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W \le 0, & 0 \le \eta \le 1, \quad t \le T, \\ W(\eta, t) - \max(\alpha - \eta, 0) \ge 0, & 0 \le \eta \le 1, \quad t \le T, \\ W(\eta, T) = \max(\alpha - \eta, 0), & 0 \le \eta \le 1, \\ \dfrac{\partial W}{\partial \eta}(1, t) = 0, & t \le T, \end{cases}$$

where we assume that $0 < \alpha \le 1$ and the operator $\mathbf{L}_\eta$ is defined by

$$\mathbf{L}_\eta \equiv \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + (D_0 - r)\,\eta\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem with an initial condition, and design a second-order implicit method for solving the new problem. (You do not need to discuss how to solve the system of linear equations.)

5. (20 %) In the Cox-Ross-Rubinstein binomial method for European options,

$$\begin{aligned} & V(S_n, n\Delta t) \\ = \; & e^{-r\Delta t}\left[pV\left(S_{n+1,1},(n+1)\Delta t\right)+(1-p)V\left(S_{n+1,0},(n+1)\Delta t\right)\right], \end{aligned}$$

where

$$p=\frac{e^{(r-D_0)\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}, \quad S_{n+1,0}=S_n e^{-\sigma\sqrt{\Delta t}}, \quad \text{and} \quad S_{n+1,1}=S_n e^{\sigma\sqrt{\Delta t}}.$$

Show that it almost is an explicit difference scheme for the following equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial y^2}+\left(r-D_0-\frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial y}-rV=0,$$

where $y=\ln S$. ("Almost" means that they have the same mesh and the differences between the coefficients of the two methods are $O(\Delta t^{3/2})$.)

6. (15%) Suppose that for $V(S,t)$ there is the following jump condition:

$$V(S,t_i^-)=V(S-D_i(S),t_i^+).$$

$V_0(S,t)$ is a smooth function. Let

$$\begin{cases} \xi=\dfrac{S}{S+P_m}, \\ \tau=T-t, \\ \overline{V}(\xi,\tau)=\dfrac{V(S,t)-V_0(S,t)}{S+P_m}, \\ \overline{V_0}(\xi,\tau)=\dfrac{V_0(S,t)}{S+P_m}, \end{cases}$$

where P_m is a positive constant. Derive the jump condition for the function $\overline{V}(\xi,\tau)$ and write this condition in a form where only $\overline{V}$ and $\overline{V_0}$ appear, i.e., V_0 does not appear.