MATH 6204 **Test I** **Spring 2007**

Name :______________________

ID :________________________

Show all the details of your work !!

1. (3.5 points) Suppose $f(x) = 0$ is a nonlinear equation. Derive the iteration formulae of Newton's method and the secant method for solving the nonlinear equation.

2. (7 points) Design a direct method for solving the following almost tridiagonal system
$$\mathbf{A}\mathbf{x} = \mathbf{q},$$
where
$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & 0 & & \\ & \ddots & \ddots & \ddots & & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} & \\ & & a_m & b_m & c_m & \end{bmatrix}.$$
(Give a complete description of the method.)

3. (7 points) For the scheme with variable coefficients
$$\begin{aligned} &\frac{u_m^{n+1} - u_m^n}{\Delta\tau} \\ = \; &\frac{1}{4}[x_m(1-x_m)\bar{\sigma}_m]^2 \left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right) \\ &+ \frac{1}{2}(r - D_0)x_m(1-x_m)\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right) \\ &- \frac{1}{2}\left[r\left(1-x_m\right) + D_0 x_m\right]\left(u_m^{n+1} + u_m^n\right), \end{aligned}$$
show that the condition $|\lambda_\theta(x_m, \tau^n)| \le 1 + O(\Delta\tau)$ is satisfied for any $x \in [0, 1]$.

4. (3.5 points) The heat equation
$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial x^2}$$

can be approximated by the explicit first-order scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}$$

or the implicit second-order scheme (the Crank–Nicolson scheme)

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right).$$

When do we choose the explicit first-order scheme and when do we use the implicit second-order scheme? Why should we choose the implicit second-order scheme if we need highly accurate results?

5. (7 points) As we know, an American lookback strike put option is the solution of the following linear complementarity problem:

$$\begin{cases} \left(\dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W\right) \cdot \left[W(\eta, t) - \max\left(\eta - \beta, 0\right)\right] = 0, & 1 \le \eta, \quad t \le T, \\ \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W \le 0, & 1 \le \eta, \quad t \le T, \\ W(\eta, t) - \max\left(\eta - \beta, 0\right) \ge 0, & 1 \le \eta, \quad t \le T, \\ W\left(\eta, T\right) = \max\left(\eta - \beta, 0\right), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}\left(1, t\right) = 0, & t \le T, \end{cases}$$

where we assume that $\beta \ge 1$ and the operator $\mathbf{L}_\eta$ is defined by

$$\mathbf{L}_\eta \equiv \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + \left(D_0 - r\right)\eta\frac{\partial}{\partial\eta} - D_0.$$

Convert this problem into a problem on $[0, 1]$ and with an initial condition, and design a second-order implicit method for solving the new problem. (You do not need to discuss how to solve the system of finite difference equations.)

6. (7 points) Suppose that for $V(S, t)$ there is the following jump condition:

$$V(S, t_i^-) = V(S - D_i(S), t_i^+).$$

$V_0(S,t)$ is a smooth function. Let

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ \overline{V}(\xi,\tau) = \dfrac{V(S,t) - V_0(S,t)}{S+P_m}, \\ \overline{V_0}(\xi,\tau) = \dfrac{V_0(S,t)}{S+P_m}, \end{cases}$$

where P_m is a positive constant. Derive the jump condition for the function $\overline{V}(\xi,\tau)$ and write this condition in a form where only $\overline{V}$ and $\overline{V_0}$ appear, i.e., V and V_0 do not appear explicitly.

7. (Bonus problem, extra 1.5 points) Suppose that we solve a problem by using a second order scheme. We already have the result on a 20×24 mesh and we want to use extrapolation technique to get a result with an error of $O(\Delta\tau^3)$.

   (a) What is the coarsest mesh which can be used for this purpose and is finer than the 20×24 mesh?

   (b) Find the value of d for this case in the extrapolation formula.