MATH 6204 **Test II** **Spring 2006**

Name :____________________
ID :____________________

Show all the details of your work !!

1. Consider the following free-boundary problem related to one-factor convertible bonds:

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, & \\ & 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), & 0 \le S \le S_f(T), \\ B_c\left(S_f(t), t\right) = nS_f(t), & 0 \le t \le T, \\ \dfrac{\partial B_c}{\partial S}\left(S_f(t), t\right) = n, & 0 \le t \le T, \\ S_f(T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right). & \end{cases}$$

   Convert this problem into a problem whose solution has a continuous derivative everywhere, and which is defined on $[0,1] \times [0,T]$ and has an initial condition.

2. Consider the following free-boundary problem:

$$\begin{cases} \dfrac{\partial V}{\partial \tau} = \dfrac{1}{2}\sigma^2 \xi^2 (1-\xi)^2 \dfrac{\partial^2 V}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial V}{\partial \xi} & \\ \qquad -[r(1-\xi) + D_0 \xi]V, & 0 \le \xi < \xi_f(\tau), \quad 0 \le \tau, \\ V(\xi, 0) = \max(2\xi - 1, 0), & 0 \le \xi < \xi_f(0), \\ V\left(\xi_f(\tau), \tau\right) = 2\xi_f(\tau) - 1, & 0 \le \tau, \\ \dfrac{\partial V}{\partial \xi}\left(\xi_f(\tau), \tau\right) = 2, & 0 \le \tau, \\ \xi_f(0) = \max\left(\dfrac{1}{2}, \dfrac{r}{r + D_0}\right). & \end{cases}$$

   Discretizing this problem by a second-order finite-difference scheme.

3. Is the extrapolation technique always helpful and why?

4. Consider the following problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathbf{L}_r^* U, & r_l \le r \le r_u, \quad 0 \le t, \\ U(r,0) = \delta(r - r^*), & r_l \le r \le r_u, \\ U(r_l, t) = 0, & 0 \le t, \\ U(r_u, t) = 0, & 0 \le t, \end{cases}$$

where

$$\mathbf{L}_r^* = \frac{\partial}{\partial r}\left(\frac{1}{2}w^2\frac{\partial}{\partial r}\right) - \frac{\partial}{\partial r}\left\{\left[u - \left(\lambda(t) + \frac{\partial w}{\partial r}\right)w\right]\right\} - r.$$

Here $\lambda(t)$ is a unknown function. We want to find such a function $\lambda(t)$ that

$$\int_{r_l}^{r_u} U(r,t)dr = f(t) \qquad \text{for any } t \in [0, T^*_{max}],$$

where $f(t)$ is a given function. Design a second-order implicit finite-difference method for such a purpose.

5. Describe how to find the market price of risk for r by using the bond equation directly and taking the prices of today's zero-coupon bonds with various maturities as input.

6. How is the final value of a swaption determined if a one-factor interest rate model is used and if the three-factor interest rate model is used. (Hint: The value of a swap is given by

$$Q\left[1 - \frac{r_{se}}{2}\sum_{k=1}^{2N} Z\left(T; T + k/2\right) - Z\left(T; T + N\right)\right],$$

where $Z\left(T; T + k/2\right)$ is the value of the $\dfrac{k}{2}$-year zero-coupon bond at time T.)