MATH 6204 **Test II** **Spring 2007**

Name :____________________

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Show all the details of your work !!

1. Consider the following free-boundary problem related to one-factor convertible bonds:

$$
\begin{cases}
\dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, & \\
 & 0 \le S \le S_f(t), \quad 0 \le t \le T, \\
B_c(S,T) = \max(Z, nS), & 0 \le S \le S_f(T), \\
B_c\left(S_f(t), t\right) = nS_f(t), & 0 \le t \le T, \\
\dfrac{\partial B_c}{\partial S}\left(S_f(t), t\right) = n, & 0 \le t \le T, \\
S_f(T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right). &
\end{cases}
$$

Convert this problem into a problem whose solution has a continuous derivative everywhere, and which is defined on $[0,1] \times [0,T]$ and has an initial condition.

2. Consider the nonlinear system consisting of the following equations

$$
\begin{aligned}
& \frac{u_m^{n+1} - u_m^n}{\Delta \tau} \\
= \ & \frac{1}{2}\left[k_2 m^2 \left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) + \frac{k_1 m}{2}\left(u_{m+1}^{n+1} - u_{m-1}^{n+1}\right) - k_0 u_m^{n+1}\right] \\
& + \frac{1}{2}\left[k_2 m^2 \left(u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n}\right) + \frac{k_1 m}{2}\left(u_{m+1}^{n} - u_{m-1}^{n}\right) - k_0 u_m^{n}\right] \\
& + \frac{s_f^{n+1} - s_f^n}{\left(s_f^{n+1} + s_f^n\right)\Delta \tau}\left[\frac{m}{2}\left(u_{m+1}^{n+1} - u_{m-1}^{n+1}\right) + \frac{m}{2}\left(u_{m+1}^{n} - u_{m-1}^{n}\right)\right], \\
& \qquad m = 0, 1, 2, \cdots, M-1,
\end{aligned}
$$

and

$$u_{M}^{n+1} = g(s_f^{n+1}, \tau^{n+1}),$$

$$\frac{3u_{M}^{n+1} - 4u_{M-1}^{n+1} + u_{M-2}^{n+1}}{2\Delta\xi} = h\left(s_f^{n+1}, \tau^{n+1}\right),$$

where u_m^n are known, τ^{n+1} is given, k_0, k_1, and k_2 are constants, and $g(s,\tau)$ and $h(s,\tau)$ are given functions. Discuss how to solve this system.

3. Suppose that the equation

$$a(\xi_i)\lambda^2 + b(\xi_i)\lambda + c(\xi_i) = 0$$

has two real distinct roots $\lambda_{1,i}$ and $\lambda_{2,i}$. Define $\varphi(\xi)$ and $\varphi(\xi)$ as follows:

$$\varphi(\xi) = e^{\lambda_{1,i}(\xi-\xi_i)}, \quad \psi(\xi) = e^{\lambda_{2,i}(\xi-\xi_i)}.$$

Let $\xi_{i-1} < \xi_i < \xi_{i+1}$ and W_{i-1}, W_i, W_{i+1} be the values of $W(\xi)$ at $\xi = \xi_{i-1}, \xi_i, \xi_{i+1}$. Assume that on $[\xi_{i-1}, \xi_{i+1}]$,

$$W(\xi) = \alpha_i\varphi(\xi) + \beta_i\psi(\xi) + \gamma_i.$$

It is clear that the constants α_i, β_i, γ_i, and

$$a(\xi_i)\frac{d^2W(\xi_i)}{d\xi^2} + b(\xi_i)\frac{dW(\xi_i)}{d\xi} + c(\xi_i)W(\xi_i)$$

can be expressed in the form of linear combination of W_{i-1}, W_i and W_{i+1}. Find such concrete expressions for them.

4. Define

$$\mathbf{L}_r = \frac{\partial}{\partial r}\left[f_1(r,t)\frac{\partial}{\partial r}\right] - f_2(r,t)\frac{\partial}{\partial r} + f_3(r,t).$$

   (a) Find an operator $\mathbf{L}_r^*$ such that

$$\int_{r_l}^{r_u} \mathbf{L}_r V\, U dr = \int_{r_l}^{r_u} \mathbf{L}_r^* U\, V dr + \left[f_1\left(U\frac{\partial V}{\partial r} - V\frac{\partial U}{\partial r}\right) - f_2 V U\right]\Bigg|_{r_l}^{r_u}.$$

   This operator is called the conjugate operator of $\mathbf{L}_r$.

   (b) Suppose

$$\frac{\partial V}{\partial t} = -\mathbf{L}_r V, \quad \frac{\partial U}{\partial t} = \mathbf{L}_r^* U,$$

$$f_1(r_l,t) = f_1(r_u,t) = 0, \quad f_2(r_l,t) < 0, \quad f_2(r_u,t) > 0$$

and

$$U(r_l, t) = U(r_u, t) = 0.$$

Show

$$\int_{r_l}^{r_u} U(r,t)V(r,t)dr = constant.$$

(c) Let $U(r, 0) = \delta(r - r^*)$ and $V(r, T^*) = 1$. Prove that there is the following relation:

$$V(r^*, 0) = \int_{r_l}^{r_u} U(r, T^*)dr.$$

5. Consider the following problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathbf{L}_r^* U, & r_l \le r \le r_u, \quad 0 \le t \le T^*_{max}, \\ U(r, 0) = \delta(r - r^*), & r_l \le r \le r_u, \\ U(r_l, t) = 0, & 0 \le t \le T^*_{max}, \\ U(r_u, t) = 0, & 0 \le t \le T^*_{max}, \end{cases}$$

where

$$\mathbf{L}_r^* = \frac{\partial}{\partial r}\left(\frac{1}{2}w^2(r)\frac{\partial}{\partial r}\right) - \frac{\partial}{\partial r}\left\{\left[u(r) - \left(\lambda(t) + \frac{\partial w(r)}{\partial r}\right)w(r)\right]\right\} - r,$$

$w(r_l) = w(r_u) = 0$, $u(r_l) > 0$, and $u(r_u) < 0$. Here $\lambda(t)$ is a unknown function. We want to find such a function $\lambda(t)$ that

$$\int_{r_l}^{r_u} U(r,t)dr = f(t) \qquad \text{for any } t \in [0, T^*_{max}],$$

where $f(t)$ is a given function. Design a second-order implicit finite-difference method for such a purpose.

6. How is the final value of a half-year option on 5-year bonds determined if a one-factor interest rate model is used and if the three-factor interest rate model is used. (For three-factor model, the independent variables, besides t, are the values of three-month, 2-year, and 5-year zero-coupon bonds.)